in [11], [26] and [12]). But nothing is known for the case where $k$ is not prime (see [12]).

Let us define for every integer $k \geqslant 3$, $(U_k(n))$ as the increasing sequence of the integers without the digit $k-1$ in their base-$k$ expansion. It is not difficult to obtain:

$$(*) \qquad \forall j \in [0, k-2] \,, \qquad U_k((k-1)n + j) = kU_k(n) + j \,.$$

If one considers the sequence of first differences of, say $U_3$, one obtains the sequence:

$$1 \quad 2 \quad 1 \quad 5 \quad 1 \quad 2 \quad 1 \quad 14 \quad 1 \quad 2 \quad 1 \quad 5 \quad 1 \quad 2 \quad 1 \quad \cdots$$

This sequence resembles somewhat the paperfolding sequence, (except that it takes infinitely many values), which gives the idea of the following easy proposition:

PROPOSITION. *Let $k$ be an integer greater than or equal to 3, define the sequence $(U_k(n))$ by $(*)$. Let $D_k(n) = U_k(n+1) - U_k(n)$. Finally let $g_k$ be defined on $\mathbf{N} \cup \{\omega\}$ by $g_k(x) = kx - k + 2$ if $x$ is in $\mathbf{N}$ and $g_k(\omega) = \omega$.*

*Then*

$$D_k = Tt((1^{k-2}\omega)^\infty, g_k) \,,$$

*(see notations in paragraph 1).*

*Proof.* From the definition of $U_k$, one has

$$D_k((k-1)n + j) = 1 \qquad \text{for every } j \text{ in } [0, k-3] \text{ and every integer } n \,,$$

$$D_k((k-1)n + k - 2) = kD_k(n) - (k-2) = g_k(D_k(n)) \qquad \text{for every integer } n \,.$$

*Remark.* For a very curious occurrence of the sequence $U_k$ see [19].

## 6. MISCELLANEOUS QUESTIONS

In this paragraph we first give some other examples of naturally occurring Toeplitz sequences. Second we shall study the connections with automatic sequences.

1) Among other examples of Toeplitz transforms let us give three natural sequences:

— Let $p$ be a prime number, and $v_p(n)$ be the highest power of $p$ dividing $n$. Let $U(n) = v_p(n+1)$, and let $f$ be the function defined over $\mathbf{N} \cup \{\omega\}$ by $f(x) = x + 1$ for every integer $x$ and $f(\omega) = \omega$. Then: $U = Tt((0^{p-1}\omega)^{\infty}, f)$.

— Define, for $n \geqslant 1$, $Q(n) = 1$ if $n$ is the sum of three squares, and $Q(n) = 0$ otherwise. Then $Q = Tt((111\omega\, 110\omega)^{\infty}, id)$.

— Let $C(n)$ be the van der Corput sequence (see [8]), used in the theory of distribution modulo 1 and defined by:

$$\text{if } n = \sum_{i \geqslant 0} b_i(n) 2^i, \text{ where } b_i \text{ is 0 or 1, then } C(n) = \sum_{i \geqslant 0} b_i(n) 2^{-i-1}.$$

Let $V(n) = C(n+1) - C(n)$ be the difference sequence of $C$. Let finally $h$ be defined on the rational numbers by $h(x) = \dfrac{x-1}{2}$ and at $\omega$ by $h(\omega) = \omega$.

Then one has $V = Tt\left(\left(\frac{1}{2}\,\omega\right)^{\infty}, h\right)$.

The first and second proofs are left to the reader; a hint for the third proof is that $C(2n) = \dfrac{C(n)}{2}$ and $C(2n+1) = \dfrac{(1 + C(n))}{2}$.

2) The interested reader can find in [27] a beautiful continued fraction expansion for $\psi(1)$ where $\psi$ is the Carlitz exponential function for $F_2[T]$, and he will certainly recognize a Toeplitz transform hidden in this expansion.

3) Actually all sequences given so far are either $q$-automatic (see [6] or [2]) or $q$-regular (see [5]). Let us recall that a sequence $(U(n))$ is said to be $q$-automatic if its $q$-kernel (i.e. the set of subsequences $n \to U(q^k n + r)$ of the sequence $U$, where $k \geqslant 0$ and $0 \leqslant r \leqslant q^k - 1$) is finite. A sequence $U$ with values in a noetherian ring $R$ is said to be $q$-regular if its $q$-kernel spans an $R$-module of finite type.

If one takes the regular paperfolding sequence $A$, it is not hard to check that ist 2-kernel is finite and equal to $\{A, (01)^{\infty}, (0)^{\infty}, (1)^{\infty}\}$, hence the sequence $A$ is 2-automatic.

In which case does a periodic sequence $B$ with values in a finite alphabet give rise to an automatic Toeplitz transform? We give the following answer to this question:

THEOREM. *Let $B$ be a periodic sequence of period $T$ with values in $\Gamma = \{a_1, a_2, \cdots, a_r, \omega\}$, such that $B(0) \neq \omega$. Denote by $d$ the cardinality of the set $\{h \in [0, T-1] \mid B(h) = \omega\}$. If $d \geqslant 1$ and $d$ divides $T$, then $Tt(B, id)$ is $(T/d)$-automatic.*

*Proof.* Let $A = Tt(B, id)$. Denote by $h_0 < h_1 < h_2 < \cdots$ the strictly increasing sequence of the integers to which $B$ assigns $\omega$. Let $h_0 < h_1 < \cdots < h_{d-1}$ be the values of $h_j$ which belong to $[0, T-1]$. It readily follows from the definition of $A$ that $A$ can be recursively defined by

$$\forall n \notin \{h_0, h_1, \cdots\} \quad A(n) = B(n) ,$$
$$\forall n \geqslant 0 , \quad A(h_n) = A(n) .$$

Moreover it is not hard to check that

$$\forall n \geqslant 0 , \quad \forall j \in [0, d-1] , \quad h_{dn+j} = h_j + Tn .$$

Hence

$$\forall n \geqslant 0 , \quad \forall j \in [0, d-1] , \quad \forall a \in [0, d-1] - \{h_0, h_1, \cdots, h_{d-1}\} ,$$
$$A(h_j + Tn) = A(h_{dn+j}) = A(dn+j) ,$$
$$A(a + Tn) = B(a + Tn) = B(a) .$$

Now let us define the set of sequences $S$ with values in $\Gamma - \{\omega\}$ by: $U$ is an element of $S$ if and only if for every $j = 0, 1, \cdots, d-1$, the sequence $n \to U(dn+j)$ is either constant (hence the constant is an element of $\Gamma - \{\omega\}$), or of the form $n \to A(dn+k)$ for some $k = k(j) \in [0, d-1]$. Notice that the set $S$ is finite as it contains at most $(\mathrm{Card}\,\Gamma - 1 + d)^d$ elements. Let $e = T/d$. To prove that the $e$-kernel of the sequence $A$ is finite, it suffices to prove that the set $S$ is stable under the maps $(X(n)) \to (X(en + r))$ for every $r$ in $[0, e-1]$ (note that the sequences $n \to A(q^k n + r)$, $k \geqslant 0, 0 \leqslant r \leqslant q^k - 1$ are obtained from the sequence $n \to A(n)$ by applying finitely many such maps, and that the sequence $A$ itself belongs to $S$).

So let us take a sequence $X$ in the set $S$, and an element $r$ in $[0, e-1]$. Let $W(n) = X(en + r)$. To prove that $W$ is in $S$, one computes $W(dn+j)$ for every $j$ in $[0, d-1]$: $W(dn+j) = X(e(dn+j)+r) = X(d(en) + ej + r)$, (note that $ej + r \leqslant e(d-1) + e - 1 = T - 1$). Define $a$ and $b$ by $ej + r = ad + b$, with $b$ in $[0, d-1]$, (hence $a$ is in $[0, e-1]$). One has

$$W(dn+j) = X(den + ej + r) = X(d(en+a)+b) .$$

As $X$ belongs to $S$, the sequence $(X(dn+b))$ is either constant or equal to the sequence $(A(dn+c))$ for a certain $c$ in $[0, d-1]$. Hence $(X(d(en+a)+b))$ is either constant or equal to the sequence $(A(d(en+a)+c)) = (A(Tn + ad + c))$. But in turn the sequence $(A(Tn + ad + c))$ is either constant (if $ad + c$ is not one of the $h_j$'s), or equal to the sequence $(A(Tn + h_u)) = (A(dn+u))$ (if $ad + c = h_u$, hence $u \leqslant d-1$ because $ad + c \leqslant T - 1$).

Finally one has $(W(dn+j)) = (X(den+ej+r)) = (X(d(en+a)+b))$ is either constant or equal to the sequence $(A(dn+u))$, hence $W$ belongs to $S$, which concludes the proof.

4) One can notice that the definition of $Tt(B, id)$ can be rewritten without supposing that $B$ is periodic. One can then ask whether the Toeplitz transform of an automatic sequence is still automatic (see [1] for a particular case):

PROPOSITION. *Let $B$ be a (non necessarily periodic) sequence on the alphabet $\Gamma$ such that the set of $h$ for which $B(h) = \omega$ is exactly the set $\{qn + b;\ n \geqslant 0\}$, where $q$ and $b$ are two natural numbers ($q \geqslant 2$ and $0 < b < q$).*

*Then $Tt(B, id)$ is $q$-automatic if and only if $B$ is itself $q$-automatic.*

*If the set $B^{-1}(\omega)$ is not of the previous kind, the result need not hold.*

*Proof.* Let $A = Tt(B, id)$. Then, as usual,

$$\forall n \notin \{qk + b\ ;\ k \geqslant 0\} \qquad A(n) = B(n)\,,$$
$$\forall n \geqslant 0 \qquad A(qn + b) = A(n)\,.$$

From the first relation one has

$$\forall j \in [0, q-1] - \{b\}\,, \quad A(qn+j) = B(qn+j)\,.$$

Moreover $B(qn + b) = \omega$.

If $A$ is $q$-automatic, so are the sequences $(A(qn+j))$, hence so are all the sequences $(B(qn+j))$ for $j$ in $[0, q-1] - \{b\}$; but the sequence $(B(qn+b))$ is constant, hence $q$-automatic. Finally all the sequences $(B(qn+j))$ are $q$-automatic for $j$ in $[0, q-1]$, and this implies the $q$-automaticity of the sequence $B$ itself.

If $B$ is $q$-automatic, let $K$ be its $q$-kernel (remember this is the set of subsequences $\{n \to B(q^k n + r);\ k \geqslant 0,\ 0 \leqslant r \leqslant q^k - 1\}$). $K$ is finite. It is clear that the $q$-kernel of $A$ is included in $K \cup \{A\}$, hence also finite, thus $A$ is a $q$-automatic sequence.

Finally we give an example of a 2-automatic sequence which does not satisfy the condition on $B^{-1}(\omega)$, and for which $Tt(B, id)$ is not 2-automatic. Indeed define $B$ by:

$$B(2^n) = \omega \quad \forall n \geqslant 1,$$
$$B(k) = 1 \quad \text{if } k \text{ is odd},$$
$$B(k) = 0 \quad \text{if } k \text{ is even and not in } \{2, 4, 8, 16, \cdots\}.$$

One easily computes

$$\begin{array}{lll} B(2n) &= \omega & \text{if } n \in \{1, 2, 4, 8, \cdots\}, \\ B(2n) &= 0 & \text{otherwise}, \\ B(2n+1) &= 1 & \forall n \geqslant 0, \\ B(4n) &= B(2n) & \forall n \geqslant 0, \\ B(4n+2) &= \omega & \text{if } n = 0, \\ B(4n+2) &= 0 & \forall n \geqslant 1, \\ B(8n+2) &= B(4n+2) & \forall n \geqslant 0, \\ B(8n+6) &= 0 & \forall n \geqslant 0. \end{array}$$

Hence the sequence $B$ is 2-automatic (its kernel contains 5 elements). Note that $A = Tt(B, id)$ satisfies

$$\begin{array}{lll} A(n) &= 1 & \text{if } n \text{ is odd}, \\ A(2^n) &= A(n) & \forall n \geqslant 1, \\ A(n) &= 0 & \text{if } n \text{ is even and not in } \{2, 4, 8, \cdots\}. \end{array}$$

If $A$ were a 2-automatic sequence, it is well known that the sequence $A(2^n)$ would be ultimately periodic; hence from the second relation the sequence $A$ itself would be ultimately periodic. Taking $j$ large enough, and looking at $A(2^j+1), A(2^j+2), \cdots, A(2^{j+1}-1)$, one sees that $A$ ends ultimately by $010101\cdots$ (or by $101010\cdots$, which is the same!). But for a huge odd number $u$ one has $A(2^u) = A(u) = 1$, and $A(2^u+1) = 1$, which yields the desired contradiction.

*Remark.* A recent paper studies the ergodic properties of the generalized Rudin-Shapiro sequences (in the sense of [6]) using the Toeplitz device: A criterion for Toeplitz flows to be topologically isomorphic and applications, J. Kwiatkowski and Y. Lacroix, preprint, 1991.

*Acknowledgements.* This work was done while the first author was visiting the Université de Genève, supported by a grant of the Fonds National de la Recherche Scientifique. The first author wants to thank all the members of the Math. Department, especially P. de la Harpe and M. Kervaire for many interesting discussions, and T. Vust for helpful suggestions.