§3. Homology of the group of piecewise linear homeomorphisms

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.07.2024**

This foliation restricted to $T^{n-1} \times [-1, 1]$ induces a foliation of $T^i * T^{n-1-i}$ which is

$$T^{n-1} \times [-1,1]/(T^i \times T^{n-1-i} \times \{-1\} \sim T^i \times \{-1\}, \\ T^i \times T^{n-1-i} \times \{1\} \sim T^{n-1-i} \times \{1\}) .$$

Note that there is a degree one map from the suspension of T^{n-1} to $T^i * T^{n-1-i}$. Since we can embed $T^{n-1} \times [-1, 1]$ in S^n, we have a degree one map from S^n to the suspension of T^{n-1}, hence to $T^i * T^{n-1-i}$. Thus Hurewicz map is surjective.

§3. Homology of the group of piecewise linear homeomorphisms

Let $PL_c(\mathbf{R})$ denote the group of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support. Let $\mu : PL_c(\mathbf{R}) \times PL_c(\mathbf{R}) \to PL_c(\mathbf{R})$ be the composition of two isomorphisms $PL_c(\mathbf{R}) \cong PL_c((-\infty, 0))$ and $PL_c(\mathbf{R}) \cong PL_c((0, \infty))$, and the inclusion

$$PL_c((-\infty, 0)) \times PL_c((0, \infty)) \to PL_c(\mathbf{R}) .$$

Then μ induces a product $*$ on the homology of $BPL_c(\mathbf{R})^\delta$ ([10]).

The homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support is described as follows. For positive integers i and j, put

$$V^{i,j} = \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge j} = \overbrace{(\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R})}^{i} \otimes_{\mathbf{Q}} \overbrace{(\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R})}^{j} .$$

Theorem (3.1).

$$H_m(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong \sum V^{k_1^-, k_1^+} \otimes_{\mathbf{Q}} \dots \otimes_{\mathbf{Q}} V^{k_s^-, k_s^+} ,$$

where the sum is taken over even number of positive integers

$$(k_1^-, k_1^+, \dots, k_s^-, k_s^+)$$

such that $k_1^- + k_1^+ + \dots + k_s^- + k_s^+ = m$. *Moreover, the $*$-product*

$$* : H_i(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \times H_j(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \to H_{i+j}(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$$

coincides with the tensor product.

For small dimensions, this theorem says that

$$H_1(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong 0 ,$$
$$H_2(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} ,$$
$$H_3(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \otimes_{\mathbf{Q}} \mathbf{R} \oplus \mathbf{R} \otimes_{\mathbf{Q}} (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}), \text{ and}$$
$$H_4(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \otimes_{\mathbf{Q}} \mathbf{R} \oplus (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \otimes_{\mathbf{Q}} (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \oplus \mathbf{R} \otimes_{\mathbf{Q}} (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \oplus \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} .$$

The first homology group is 0 is equivalent to that $PL_c(\mathbf{R})$ is perfect and this is due to Epstein ([1]). The second and third homologies are given explicitly by Greenberg ([7]). The summand $\mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$ of $H_4(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ is the image of the $*$-product on $H_2(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ and since the $*$-product coincides with the tensor product, the $*$-product is not graded commutative. This implies that the Whitehead product

$$\pi_3(B\bar{\Gamma}_1^{PL}) \times \pi_3(B\bar{\Gamma}_1^{PL}) \to \pi_5(B\bar{\Gamma}_1^{PL})$$

is highly nontrivial and this is the obstruction to construct a foliation on $S^3 \times S^3$ with given Godbillon-Vey class. In this way, as in mentioned in §2, this is related to the rationality (see [10]).

Theorem (3.1) is also obtained as an application of the description by Greenberg ([7]) of the classifying space $B\bar{\Gamma}_1^{PL}$. As we mentioned, his result says that this classifying space is weakly homotopy equivalent to the join $B\mathbf{R}^\delta * B\mathbf{R}^\delta$. To show Theorem (3.1), we use the isomorphism

$$H_*(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong H_*(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z})$$

due to Mather ([9]) adapted for the PL case by Ghys-Sergiescu ([5]) and Greenberg ([7]) using a result of Segal ([11]), and the homology spectral sequence associated to the fibration

$$\Omega B\bar{\Gamma}_1^{PL} \to PB\bar{\Gamma}_1^{PL} \to B\bar{\Gamma}_1^{PL} .$$

Since $B\bar{\Gamma}_1^{PL}$ is simply connected, the E^2 term of this spectral sequence is as follows.

$$E^2_{p+1,q} = H_{p+1}(B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z}) .$$

Note that $H_*(B\bar{\Gamma}_1^{PL}; \mathbf{Z})$ is torsion free. From this, Greenberg obtained the second and the third homologies ([7]). To show our theorem, we show that, for $p \geqslant 0$,

$$E^2_{p+1,q} = Z^2_{p+1,q} = \dots = Z^p_{p+1,q} \quad \text{and}$$
$$Z^{p+1}_{p+1,q} = Z^\infty_{p+1,q} = 0 = B^\infty_{p+1,q} = \dots = B^2_{p+1,q} .$$

This is equivalent to that the differentials induce an isomorphism

$$\sum_{p+q=m, p\geqslant 0} H_{p+1}(B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \to H_{p+q}(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z}) .$$

To show this we define the cohomology classes of $BPL_c(\mathbf{R})^{\delta}$ which detect the images of generators of $H_{p+1}(B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z})$.

§4. Construction of cocycles of the group $PL_c(\mathbf{R})$

Tensor determinants. We define a determinant of an $(n \times n)$ real matrix which takes values in the tensor product over $\mathbf{Q}$ of n copies of $\mathbf{R}$. For $(a_{ij})_{i,j=1,\ldots,n}$, we put

$$\det{}^{\otimes_{\mathbf{Q}}}(a_{ij}) = \sum_{\sigma} \operatorname{sign}(\sigma)\, a_{\sigma(1)1} \otimes_{\mathbf{Q}} \cdots \otimes_{\mathbf{Q}} a_{\sigma(n)n} .$$

For example,

$$\det{}^{\otimes_{\mathbf{Q}}} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \otimes_{\mathbf{Q}} a_{22} - a_{21} \otimes_{\mathbf{Q}} a_{12} .$$

We have the usual multilinearity but we do not have the usual alternativity. For example,

$$\det{}^{\otimes_{\mathbf{Q}}} \begin{pmatrix} a & b \\ a & b \end{pmatrix} = 0 \quad \text{but} \quad \det{}^{\otimes_{\mathbf{Q}}} \begin{pmatrix} a & a \\ b & b \end{pmatrix} = a \otimes_{\mathbf{Q}} b - b \otimes_{\mathbf{Q}} a = a \wedge_{\mathbf{Q}} b .$$

The latter is not necessarily zero. In general, if we change the rows then this determinant changes sign, however, there are no simple laws for changing columns. It is worth noticing that we have the usual formula of developing with respect to the first or the last column.

$$\begin{aligned} \det{}^{\otimes_{\mathbf{Q}}}(a_{ij}) &= \sum_{i+1}^{n} (-1)^{i+1} a_{i1} \otimes_{\mathbf{Q}} \det{}^{\otimes_{\mathbf{Q}}}(A_{i1}) \\ &= \sum_{i+1}^{n} (-1)^{i+n} \det{}^{\otimes_{\mathbf{Q}}}(A_{in}) \otimes_{\mathbf{Q}} a_{in} , \end{aligned}$$

where A_{ij} is the matrix (a_{ij}) with the i-th row and the j-th column deleted.

Cocycles of Lipschitz homeomorphism groups. We review the construction of cocycles of certain Lipschitz homeomorphism groups of the real line or the circle (see [13]). Let $\mathscr{S}$ be the space of functions with compact support