

# 5. COMMENTS ON THE EXAMPLES IN TABLES II

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The first equation gives

$$P\bar{P} + Q\bar{Q} = 11 + 11(1 + U + \bar{U})(1 + V + \bar{V}).$$

Substituting in the second equation, we get

$$(*) \quad P + \bar{Q} + 5\bar{P}Q + 4P\bar{Q} = -44 + 55(1 + U + \bar{U})(1 + V + \bar{V}).$$

Since  $U\bar{U} = 1$ ,  $U^2 = \bar{U}$  and  $V\bar{V} = 3 + V + \bar{V}$ ,  $V^2 = V + 2\bar{V}$ , the constant terms in  $\bar{P}Q$  and  $P\bar{Q}$  are equal to  $\sum_{i=0}^2 p_i q_i + 3 \sum_{j=3}^8 p_j q_j = c$ , say. Hence, equating constant terms in the above equation (\*), we must have

$$p_0 + q_0 + 9c = 11.$$

The only solution to this equation with all  $p_i, q_i$  being 0 or 1, is  $p_0 = q_0 = 1$ ,  $p_i = q_i = 0$  for  $i = 1, \dots, 8$ . This means  $P = Q = 1$ , contradicting (\*).

## 5. COMMENTS ON THE EXAMPLES IN TABLES II

Difference sets with parameters  $(v, k, \lambda) = (4n - 1, 2n - 1, n - 1)$  are usually called *Hadamard difference sets*. Our purpose here is to discuss the classification of these cyclic difference sets for  $2 \leq n \leq 100$ .

In many cases where  $v = 4n - 1$  is a prime  $p$ , the quadratic residue difference set, which we denote by  $QR(p)$  is unique for the given values of the parameters. This is obviously the case if the multiplier  $m$  has order

$k = \frac{1}{2}(v - 1)$  in  $(\mathbf{Z}/v\mathbf{Z})^*$ . Indeed, in this case, there are exactly 3 orbits of

multiplication by  $m$  in  $\mathbf{Z}/v\mathbf{Z}$ , namely  $1 = \{0\}$ ,  $M = \{1, m, m^2, \dots, m^{k-1}\}$  and  $\bar{M} = \{-1, -m, \dots, -m^{k-1}\}$ . Thus the only choice for  $D$  is  $D = M$  or  $D = \bar{M}$ , which are isomorphic under conjugation  $\sigma: \mathbf{Z}/v\mathbf{Z} \rightarrow \mathbf{Z}/v\mathbf{Z}$ ,  $\sigma(a) = -a$ .

In our Table II, this situation happens for  $n = 3, 5, 6, 12, 15, 17, 18, 20, 21, 27, 33, 35, 41, 42, 45, 48, 53, 57, 60, 63, 66, 68, 77, 87, 90$  and  $96$ .

The remaining cases where  $v = 4n - 1$  is a prime  $p$  (for  $2 \leq n \leq 100$ ) have been shown to lead to a single difference set, namely  $QR(p)$ , by machine enumeration of the various choices of  $D$  as a union of orbits under multiplication by a multiplier  $m$ . This includes the cases  $n = 26$  (multiplier 8),  $n = 38$  (multiplier 19),  $n = 50$  (multiplier 5),  $n = 78$  (multiplier 13),  $n = 83$  (multiplier 83), and  $n = 95$  (multiplier 5). By far, the most difficult case (for the machine) occurs with  $n = 38$ , which required the examination of 37 442 160 combinations of multiplier orbits.

The case  $n = 36$ , also leads by machine enumeration to the single difference set  $TP(11, 13)$  of twin-prime type with parameters  $(143 = 11 \cdot 13, 71, 35)$ .

The only other values of  $n (\leq 100)$  for which  $v = 4n - 1$  is *not* a prime and a Hadamard cyclic difference set with parameters  $(4n - 1, 2n - 1, n - 1)$  does exist are powers of 2. The examples for  $n = 2, 4$  and 8 are easily seen to be unique.

For  $n = 16$ , there are 2 isomorphism classes of Hadamard difference sets with parameters  $(63, 31, 15)$ : Both have multiplier  $m = 2$ , and denoting by  $X_a$  the orbit of  $a$  under multiplication by 2 in  $\mathbf{Z}/63\mathbf{Z}$ , they are

$$D_0 = 1 + X_{-25} + X_{-9} + X_1 + X_3 + X_7 + X_9 ,$$

which is isomorphic to  $\mathbf{P}^5(\mathbf{F}_2)$ , and

$$D_1 = 1 + X_{-9} + X_{-1} + X_1 + X_3 + X_9 + X_{25} ,$$

which is of type *GMW*.

The difference sets  $D_0$  and  $D_1$  are not isomorphic, even as block designs, as can be seen by computing the cardinalities of the intersection of triples of blocks of  $D_1$ , giving the enumerating polynomial

$$10584t^6 + 19656t^7 + 3528t^8 + 5880t^9 + 63t^{15} ,$$

in contrast to  $39060t^7 + 651t^{15}$  for  $\mathbf{P}^5(\mathbf{F}_2)$ . (The coefficient of  $t^i$  being the multiplicity of triple intersections of cardinality  $i$ .)

For  $n = 32$ , L. Baumert and H. Fredricksen have found that there are exactly 6 non-isomorphic examples. (See [BF].) Three of these,  $QR(127)$ ,  $\mathbf{P}^6(\mathbf{F}_2)$  and  $MH(127)$  are members of the classical families.

For  $n = 64$ , we have found that there exist exactly 4 examples (up to isomorphism). One of them is  $\mathbf{P}^7(\mathbf{F}_2)$ , another one is of type *GMW*. The other two seem to be new.

All 4 of them have multiplier 2, which is of order 8 modulo  $v = 255$ . They all contain the union  $U$  of the multiplier orbits of length  $< 8$ , viz.

$$U = \{0\} + \{85, -85\} + \{51, 102, -51, -102\} + \{17, 34, 68, -119\} \\ + \{-17, -34, -68, 119\} .$$

Denoting by  $(a_1, \dots, a_{14})$  the union  $\sum_{i=1}^{14} X_{a_i}$  of the orbits

$$X_a = \{a, 2a, \dots, 2^7a\} ,$$

the 4 examples are of the form  $D_i = U + V_i$ , where

$$V_0 = (-19, -9, -7, -1, 1, 3, 7, 13, 19, 23, 25, 27, 37, 45),$$

$$V_1 = (-43, -27, -25, -13, -9, -5, -3, 7, 11, 13, 19, 23, 27, 43),$$

$$V_2 = (-43, -27, -23, -13, -11, -3, 1, 3, 7, 13, 15, 25, 37, 43),$$

$$V_3 = (-43, -23, -21, -11, -7, -3, 7, 9, 11, 15, 19, 25, 37, 43).$$

The difference sets  $D_2$  and  $D_3$  appear to be exotic.  $D_0$  is isomorphic to  $\mathbf{P}^7(\mathbf{F}_2)$ . Finally,  $D_1$  is of type *GMW*, and can be constructed as follows.

Let  $L = \mathbf{F}_{256}$  be the extension of degree 8 over  $F = \mathbf{F}_2$ . We will use the trace  $Tr = Tr_{L/F}: L \rightarrow F$  given by  $Tr(\gamma) = \sum_{i=0}^7 \gamma^{2^i}$ . The extension  $L/F$  is defined by the irreducible polynomial  $x^8 + x^4 + x^3 + x^2 + 1 \in \mathbf{F}[x]$ . The multiplicative group  $\mathbf{F}_{256}^*$  is generated by any root  $\alpha$  of this polynomial. The Hall polynomial  $D_0(x)$  of  $D_0$  is then given by

$$D_0(x) = \sum_{i=0}^{254} d_i x^i \in \mathbf{Z}[x]/(x^{255} - 1),$$

where

$$d_i = \begin{cases} 0 & \text{if } Tr(\alpha^i) \neq 0 \\ 1 & \text{if } Tr(\alpha^i) = 0. \end{cases}$$

Thus a block of the difference set is the hyperplane  $\ker(Tr) \subset \mathbf{F}_{256} = \mathbf{F}_2^8$ .

Under the identification

$$\mathbf{Z}/255\mathbf{Z} \rightarrow \mathbf{F}_{256}^*$$

given by  $i \mapsto \alpha^i$ , the multiplication by 2 in  $\mathbf{Z}/255\mathbf{Z}$  becomes the Frobenius automorphism in the extension  $\mathbf{F}_{256}/\mathbf{F}_2$ . The block  $\ker(Tr)$  is a union of orbits under the action of the multiplier.

In order to construct  $D_1$ , the example of type *GMW*, we need the intermediate extension  $K = \mathbf{F}_{16}$ ,  $F \subset K \subset L$ . Set  $\beta = \alpha^{17}$ , a generator of  $K^* = \mathbf{F}_{16}^*$ . Denote by  $tr = tr_{K/F}: K \rightarrow F$  the trace.

Consider the complementary polynomial  $D'_0(x) = T - D_0(x)$ , where  $T = \sum_{i=0}^{254} x^i \in \mathbf{Z}[x]/(x^{255} - 1)$ . The crucial point is to observe that  $D'_0(x)$  splits as

$$D'_0(x) = \Omega(x) \cdot \theta_0(x^{17}) \in \mathbf{Z}[x]/(x^{255} - 1),$$

where  $\theta_0(y) = \sum_{j=0}^{14} a_j y^j$  with

$$a_j = \begin{cases} 0 & \text{if } tr(\beta^j) = 0 \\ 1 & \text{if } tr(\beta^j) \neq 0, \end{cases}$$

and  $\Omega(x) = x^7 + x^{14} + \cdots + x^{246}$ . Here,

$$\theta_0(y) = y + y^2 + y^3 + y^4 + y^6 + y^8 + y^9 + y^{12}.$$

Now define

$$D'_1(x) = \Omega(x) \cdot \theta_1(x^{17}),$$

where  $\theta_1(y) = \theta_0(y^{-1})$ . Then  $D_1(x) = T - D'_1(x)$  is the Hall polynomial of the difference set  $D_1$ .

The fact that  $D_0$ ,  $D_1$ ,  $D_2$  and  $D_3$  are not isomorphic, even as block designs, can again be seen by determining the cardinalities of all triple intersections of blocks, for each  $D_i$ . Denoting by  $P_i$  the corresponding enumerating polynomial of triple intersections for  $D_i$ , we have

$$P_0 = 2720340t^{31} + 10795t^{63}$$

$$P_1 = 979200t^{29} + 823140t^{31} + 734400t^{33} + 183600t^{35} \\ + 10200t^{39} + 595t^{63}$$

$$P_2 = 9180t^{25} + 8160t^{26} + 45900t^{27} + 163200t^{28} + 342720t^{29} \\ + 514080t^{30} + 518160t^{31} + 465120t^{32} + 358020t^{33} \\ + 179520t^{34} + 81090t^{35} + 18360t^{36} + 18360t^{37} + 6120t^{38} \\ + 3145t^{39}$$

$$P_3 = 4080t^{25} + 14280t^{26} + 40800t^{27} + 142800t^{28} + 385560t^{29} \\ + 403920t^{30} + 692580t^{31} + 424320t^{32} + 352920t^{33} + 128520t^{34} \\ + 79050t^{35} + 32640t^{36} + 9180t^{37} + 12240t^{38} + 7225t^{39} + 1020t^{45}.$$

TABLE I

Case  $\gamma = +1$ :

*Non-existence of a cyclic difference set  
with parameters  $(2t(t+1) + 1, t^2, \frac{1}{2}t(t-1))$  for  $3 \leq t \leq 100$ .  
(The case  $t = 50$  is still undecided.)*

$t$	$(v, k, \lambda)$	$n = k - \lambda$	reason for non-existence
3	$(5^2, 9, 3)$	$2 \cdot 3$	$2^2 \equiv -1 \pmod{5}$
4	$(41, 16, 6)$	$2 \cdot 5$	$5^{10} \equiv -1 \pmod{41}$
5	$(61, 25, 10)$	$3 \cdot 5$	$3^5 \equiv -1 \pmod{61}$
6	$(5 \cdot 17, 36, 15)$	$3 \cdot 7$	$3^2 \equiv -1 \pmod{5}$
7	$(113, 49, 21)$	$2^2 \cdot 7$	$7^7 \equiv -1 \pmod{113}$
8	$(5 \cdot 29, 64, 28)$	$2^2 \cdot 3^2$	$2^{14} \equiv -1 \pmod{145}$
9	$(181, 81, 36)$	$3^2 \cdot 5$	$5 \equiv 3^6 \pmod{181}$ would be multiplier
10	$(13 \cdot 17, 100, 45)$	$5 \cdot 11$	$5^2 \equiv -1 \pmod{13}$
11	$(5 \cdot 53, 121, 55)$	$2 \cdot 3 \cdot 11$	$2^2 \equiv -1 \pmod{5}$
12	$(3 \cdot 13, 144, 66)$	$2 \cdot 3 \cdot 13$	$2^{78} \equiv -1 \pmod{313}$
13	$(5 \cdot 73, 169, 78)$	$7 \cdot 13$	$7^2 \equiv -1 \pmod{5}$
14	$(421, 196, 91)$	$3 \cdot 5 \cdot 7$	$5^{105} \equiv -1 \pmod{421}$

TABLE I (continued)

$t$	$(v, k, \lambda)$	$n = k - \lambda$	reason for non-existence
15	(13 · 37, 225, 105)	$2^3 \cdot 3 \cdot 5$	$5^2 \equiv -1 \pmod{13}$
16	(5 · 109, 256, 120)	$2^3 \cdot 17$	$17^2 \equiv -1 \pmod{5}$
17	(613, 289, 136)	$3^2 \cdot 17$	$17^{51} \equiv -1 \pmod{613}$
18	(5 · 137, 324, 153)	$3^2 \cdot 19$	$19 \equiv -1 \pmod{5}$
19	(761, 361, 171)	$2 \cdot 5 \cdot 19$	$2^{190} \equiv -1 \pmod{761}$
20	(29 <sup>2</sup> , 400, 190)	$2 \cdot 3 \cdot 5 \cdot 7$	$2^{14} \equiv -1 \pmod{29}$
21	(5 <sup>2</sup> · 37, 441, 210)	$3 \cdot 7 \cdot 11$	$3^2 \equiv -1 \pmod{5}$
22	(1013, 484, 231)	$11 \cdot 23$	$11^{23} \equiv -1 \pmod{1013}$
23	(5 · 13 · 17, 529, 253)	$2^2 \cdot 3 \cdot 23$	$3^2 \equiv -1 \pmod{5}$
24	(1201, 576, 276)	$2^2 \cdot 3 \cdot 5^2$	$3^{150} \equiv -1 \pmod{1201}$
25	(1301, 625, 300)	$5^2 \cdot 13$	$5^{325} \equiv -1 \pmod{1301}$
26	(5 · 281, 676, 325)	$3^3 \cdot 13$	$13^2 \equiv -1 \pmod{5}$
27	(17 · 89, 729, 351)	$2 \cdot 3^3 \cdot 7$	$2^4 \equiv -1 \pmod{17}$
28	(5 <sup>3</sup> · 13, 784, 378)	$2 \cdot 7 \cdot 9$	$2^2 \equiv -1 \pmod{5}$
29	(1741, 841, 406)	$3 \cdot 5 \cdot 29$	$3^{435} \equiv -1 \pmod{1741}$
30	(1861, 900, 435)	$3 \cdot 5 \cdot 31$	$3^{155} \equiv -1 \pmod{1861}$
31	(5 · 397, 961, 465)	$2^4 \cdot 31$	$2^{22} \equiv -1 \pmod{1985}$
32	(2113, 1024, 496)	$2^4 \cdot 3 \cdot 11$	$3^{528} \equiv -1 \pmod{2113}$
33	(5 · 449, 1089, 528)	$3 \cdot 11 \cdot 17$	$3^2 \equiv -1 \pmod{5}$
34	(2381, 1156, 561)	$5 \cdot 7 \cdot 17$	$5^{119} \equiv -1 \pmod{2381}$
35	(2521, 1225, 595)	$2 \cdot 3^2 \cdot 5 \cdot 7$	$2^{630} \equiv -1 \pmod{2521}$
36	(5 · 13 · 41, 1296, 630)	$2 \cdot 3^2 \cdot 37$	$2^2 \equiv -1 \pmod{5}$
37	(29 · 97, 1369, 666)	$19 \cdot 37$	$19^{14} \equiv -1 \pmod{29}$
38	(5 · 593, 1444, 703)	$3 \cdot 13 \cdot 19$	$3^2 \equiv -1 \pmod{5}$
39	(3121, 1521, 741)	$2^2 \cdot 3 \cdot 5 \cdot 13$	$2^{78} \equiv -1 \pmod{3121}$
40	(17 · 193, 1600, 780)	$2^2 \cdot 5 \cdot 41$	$5^8 \equiv -1 \pmod{17}$
41	(5 · 13 · 53, 1681, 820)	$3 \cdot 7 \cdot 41$	$3^2 \equiv -1 \pmod{5}$
42	(3613, 1764, 861)	$3 \cdot 7 \cdot 43$	$3^{903} \equiv -1 \pmod{3613}$
43	(5 · 757, 1849, 903)	$2 \cdot 11 \cdot 43$	$2^2 \equiv -1 \pmod{5}$
44	(17 · 233, 1936, 946)	$2 \cdot 3^2 \cdot 5 \cdot 11$	$2^4 \equiv -1 \pmod{17}$
45	(41 · 101, 2025, 990)	$3^2 \cdot 5 \cdot 23$	$5^{10} \equiv -1 \pmod{41}$
46	(5 <sup>2</sup> · 173, 2116, 1035)	$23 \cdot 47$	$23^2 \equiv -1 \pmod{5}$
47	(4513, 2209, 1081)	$2^3 \cdot 3 \cdot 47$	$3^{188} \equiv -1 \pmod{4513}$
48	(5 · 941, 2304, 1128)	$2^3 \cdot 3 \cdot 7^2$	$3^2 \equiv -1 \pmod{5}$
49	(13 <sup>2</sup> · 29, 2401, 1176)	$5^2 \cdot 7^2$	$5^2 \equiv 7^6 \pmod{4901}$ would be multipli
50	(5101, 2500, 1225)	$3 \cdot 5^2 \cdot 17$	existence unsettled
51	(5 · 1061, 2601, 1275)	$2 \cdot 3 \cdot 13 \cdot 17$	$2^2 \equiv -1 \pmod{5}$
52	(37 · 149, 2704, 1326)	$2 \cdot 13 \cdot 53$	$2^{18} \equiv -1 \pmod{37}$
53	(5 <sup>2</sup> · 229, 2809, 1378)	$3^3 \cdot 53$	$53^2 \equiv -1 \pmod{5}$
54	(13 · 457, 2916, 1431)	$3^3 \cdot 5 \cdot 11$	$5^2 \equiv -1 \pmod{13}$
55	(61 · 101, 3025, 1485)	$2^2 \cdot 5 \cdot 7 \cdot 11$	$5^{15} \equiv -1 \pmod{61}$
56	(5 · 1277, 3136, 1540)	$2^2 \cdot 3 \cdot 7 \cdot 19$	$3^2 \equiv -1 \pmod{5}$
57	(17 · 389, 3249, 1596)	$3 \cdot 19 \cdot 29$	$3^8 \equiv -1 \pmod{17}$

TABLE I (continued)

$t$	$(v, k, \lambda)$	$n = k - \lambda$	reason for non-existence
58	$(5 \cdot 37^2, 3364, 1653)$	$29 \cdot 59$	$29 \equiv -1 \pmod{5}$
59	$(73 \cdot 97, 3481, 1711)$	$2 \cdot 3 \cdot 5 \cdot 59$	$3^6 \equiv -1 \pmod{73}$
60	$(7321, 3600, 1770)$	$2 \cdot 3 \cdot 5 \cdot 61$	$2^{610} \equiv -1 \pmod{7321}$
61	$(5 \cdot 17, 3721, 1830)$	$31 \cdot 61$	$31^8 \equiv -1 \pmod{17}$
62	$(13 \cdot 601, 3844, 1891)$	$3^2 \cdot 7 \cdot 31$	$7^6 \equiv -1 \pmod{13}$
63	$(5 \cdot 1613, 3969, 1953)$	$2^5 \cdot 3^2 \cdot 7$	$7^2 \equiv -1 \pmod{5}$
64	$(53 \cdot 157, 4096, 2016)$	$2^5 \cdot 5 \cdot 13$	$5^{26} \equiv -1 \pmod{53}$
65	$(8581, 4225, 2080)$	$3 \cdot 5 \cdot 11 \cdot 13$	$3^{715} \equiv -1 \pmod{8581}$
66	$(5 \cdot 29 \cdot 61, 4356, 2145)$	$3 \cdot 11 \cdot 67$	$3^2 \equiv -1 \pmod{5}$
67	$(13 \cdot 701, 4489, 2211)$	$2 \cdot 17 \cdot 67$	$2^6 \equiv -1 \pmod{13}$
68	$(5 \cdot 1877, 4624, 2278)$	$2 \cdot 3 \cdot 17 \cdot 23$	$2^2 \equiv -1 \pmod{5}$
69	$(9661, 4761, 2346)$	$3 \cdot 5 \cdot 7 \cdot 23$	$7^{2415} \equiv -1 \pmod{9661}$
70	$(9941, 4900, 2415)$	$5 \cdot 7 \cdot 71$	$7^{2485} \equiv -1 \pmod{9941}$
71	$(5^2 \cdot 409, 5041, 2485)$	$2^2 \cdot 3^2 \cdot 71$	$2^{510} \equiv -1 \pmod{10225}$
72	$(10513, 5184, 2556)$	$2^2 \cdot 3^2 \cdot 73$	$2^{1314} \equiv -1 \pmod{10513}$
73	$(5 \cdot 2161, 5329, 2628)$	$37 \cdot 73$	$37^2 \equiv -1 \pmod{5}$
74	$(17 \cdot 653, 5476, 2701)$	$3 \cdot 5^2 \cdot 37$	$3^8 \equiv -1 \pmod{17}$
75	$(13 \cdot 877, 5625, 2775)$	$2 \cdot 3 \cdot 5^2 \cdot 19$	$2^6 \equiv -1 \pmod{13}$
76	$(5 \cdot 2341, 5776, 2850)$	$2 \cdot 7 \cdot 11 \cdot 19$	$2^2 \equiv -1 \pmod{5}$
77	$(41 \cdot 293, 5929, 2926)$	$3 \cdot 7 \cdot 11 \cdot 13$	$3^4 \equiv -1 \pmod{41}$
78	$(5^2 \cdot 17 \cdot 29, 6084, 3003)$	$3 \cdot 13 \cdot 79$	$3^2 \equiv -1 \pmod{5}$
79	$(12641, 6241, 3081)$	$2^3 \cdot 5 \cdot 79$	$5^{1580} \equiv -1 \pmod{12641}$
80	$(13 \cdot 997, 6400, 3160)$	$2^3 \cdot 3^4 \cdot 5$	$5^2 \equiv -1 \pmod{13}$
81	$(5 \cdot 2657, 6561, 3240)$	$3^4 \cdot 41$	$41^{332} \equiv -1 \pmod{2657}$
82	$(13613, 6724, 3321)$	$41 \cdot 83$	$41 \equiv 83^3 \pmod{13613}$ would be multiple
83	$(5 \cdot 2789, 6889, 3403)$	$2 \cdot 3 \cdot 7 \cdot 83$	$2^2 \equiv -1 \pmod{5}$
84	$(14281, 7056, 3486)$	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 17$	$2^{1190} \equiv -1 \pmod{14281}$
85	$(14621, 7225, 3570)$	$5 \cdot 17 \cdot 43$	$5^{3655} \equiv -1 \pmod{14621}$
86	$(5 \cdot 41 \cdot 73, 7396, 3655)$	$3 \cdot 29 \cdot 43$	$3^2 \equiv -1 \pmod{5}$
87	$(15313, 7569, 3741)$	$2^2 \cdot 3 \cdot 11 \cdot 29$	$3^{1276} \equiv -1 \pmod{15313}$
88	$(5 \cdot 13 \cdot 241, 7744, 3828)$	$2^2 \cdot 11 \cdot 89$	$89 \equiv -1 \pmod{5}$
89	$(37 \cdot 433, 7921, 3916)$	$3^2 \cdot 5 \cdot 89$	$5^{18} \equiv -1 \pmod{37}$
90	$(16381, 8100, 4005)$	$3^2 \cdot 5 \cdot 7 \cdot 13$	$13^{65} \equiv -1 \pmod{16381}$
91	$(5 \cdot 17 \cdot 197, 8281, 4095)$	$2 \cdot 7 \cdot 13 \cdot 23$	$2^2 \equiv -1 \pmod{5}$
92	$(109 \cdot 157, 8464, 4186)$	$2 \cdot 3 \cdot 23 \cdot 31$	$2^{18} \equiv -1 \pmod{109}$
93	$(5 \cdot 13 \cdot 269, 8649, 4278)$	$3 \cdot 31 \cdot 47$	$3^2 \equiv -1 \pmod{5}$
94	$(53 \cdot 337, 8836, 4371)$	$5 \cdot 19 \cdot 47$	$5^{26} \equiv -1 \pmod{53}$
95	$(17 \cdot 29 \cdot 37, 9025, 4465)$	$2^4 \cdot 3 \cdot 5 \cdot 19$	$3^8 \equiv -1 \pmod{17}$
96	$(5^3 \cdot 149, 9216, 4560)$	$2^4 \cdot 3 \cdot 97$	$3^2 \equiv -1 \pmod{5}$
97	$(19013, 9409, 4656)$	$7^2 \cdot 97$	$7^{4753} \equiv -1 \pmod{19013}$
98	$(5 \cdot 3881, 9604, 4753)$	$3^2 \cdot 7^2 \cdot 11$	$11^{97} \equiv -1 \pmod{3881}$
99	$(19801, 9801, 4851)$	$2 \cdot 3^2 \cdot 5^2 \cdot 11$	$2^{4950} \equiv -1 \pmod{19801}$
100	$(20201, 10000, 4950)$	$2 \cdot 5^2 \cdot 101$	$2^{5050} \equiv -1 \pmod{20201}$

TABLE II

Case  $\gamma = -1$ :Cyclic difference sets with parameters  $(4n - 1, 2n - 1, n - 1)$ ,  $2 \leq n \leq 100$ 

$n$	$(4n - 1, 2n - 1, n - 1)$	exists?	examples or comment to non-existence
2	(7, 3, 1)	Yes	$\mathbf{P}^2(\mathbf{F}_2) = QR(7)$
3	(11, 5, 2)	Yes	$QR(11)$
4	(3 · 5, 7, 3)	Yes	$TP(3, 5) = \mathbf{P}^3(\mathbf{F}_2)$
5	(19, 9, 4)	Yes	$QR(19)$
6	(23, 11, 5)	Yes	$QR(23)$
7	(3 <sup>3</sup> , 13, 6)	No	7 would be multiplier
8	(31, 15, 7)	Yes	$\mathbf{P}^4(\mathbf{F}_2)$ and $QR(31)$
9	(5 · 7, 17, 8)	Yes	$TP(5, 7)$
10	(3 · 13, 19, 9)	No	$2 \equiv -1 \pmod{3}$
11	(43, 21, 10)	Yes	$QR(43)$
12	(47, 23, 11)	Yes	$QR(47)$
13	(3 · 17, 25, 12)	No	$13^2 \equiv -1 \pmod{17}$
14	(5 · 11, 27, 13)	No	$2^2 \equiv -1 \pmod{5}$
15	(59, 29, 14)	Yes	$QR(59)$
16	(3 <sup>2</sup> · 7, 31, 15)	Yes	$\mathbf{P}^5(\mathbf{F}_2)$ and $GMW$
17	(67, 33, 16)	Yes	$QR(67)$
18	(71, 35, 17)	Yes	$QR(71)$
19	(3 · 5 <sup>2</sup> , 35, 18)	No	$19 \equiv -1 \pmod{5}$
20	(79, 39, 19)	Yes	$QR(79)$
21	(83, 41, 20)	Yes	$QR(83)$
22	(3 · 29, 43, 21)	No	$2 \equiv -1 \pmod{3}$
23	(7 · 13, 45, 22)	No	$23^3 \equiv -1 \pmod{13}$
24	(5 · 19, 47, 23)	No	$3^2 \equiv -1 \pmod{5}$
25	(3 <sup>2</sup> · 11, 49, 24)	No	25 would be multiplier
26	(103, 51, 25)	Yes	$QR(103)$
27	(107, 53, 26)	Yes	$QR(107)$
28	(3 · 37, 55, 27)	No	$m = 7 \equiv 2^{32}$ would be multiplier
29	(5 · 23, 57, 28)	No	$29 \equiv -1 \pmod{5}$
30	(7 · 17, 59, 29)	No	$2^4 \equiv -1 \pmod{17}$
31	(3 · 41, 61, 30)	No	$31^5 \equiv -1 \pmod{41}$
32	(127, 63, 31)	Yes	$\mathbf{P}^6(\mathbf{F}_2)$ , $QR(127)$ , $MH(127)$ , and 3 others
33	(131, 65, 32)	Yes	$QR(131)$
34	(3 <sup>3</sup> · 5, 67, 33)	No	$17 \equiv -1 \pmod{3}$
35	(139, 69, 34)	Yes	$QR(139)$
36	(11 · 13, 71, 35)	Yes	$TP(11, 13)$
37	(3 · 7 <sup>2</sup> , 73, 36)	No	37 would be multiplier
38	(151, 75, 37)	Yes	$QR(151)$
39	(5 · 31, 77, 38)	No	$3^2 \equiv -1 \pmod{5}$
40	(3 · 53, 79, 39)	No	$5 \equiv -1 \pmod{3}$



TABLE 2 (continued)

$n$	$(4n-1, 2n-1, n-1)$	exists?	examples or comment to non-existence
41	(163, 81, 40)	Yes	$QR(163)$
42	(167, 83, 41)	Yes	$QR(167)$
43	$(3^2 \cdot 19, 85, 42)$	No	43 would be multiplier
44	$(5^2 \cdot 7, 87, 43)$	No	$m = 11 \equiv 2^{56} \pmod{175}$ would be multiplier
45	(179, 89, 44)	Yes	$QR(179)$
46	$(3 \cdot 61, 91, 45)$	No	$23 \equiv -1 \pmod{3}$
47	$(11 \cdot 17, 93, 46)$	No	$47^2 \equiv -1 \pmod{17}$
48	(191, 95, 47)	Yes	$QR(191)$
49	$(3 \cdot 5 \cdot 13, 97, 48)$	No	7 would be multiplier
50	(199, 99, 49)	Yes	$QR(199)$
51	$(7 \cdot 29, 101, 50)$	No	$3^3 \equiv -1 \pmod{7}$
52	$(3^2 \cdot 23, 103, 51)$	No	13 would be multiplier
53	(211, 105, 52)	Yes	$QR(211)$
54	$(5 \cdot 43, 107, 53)$	No	$2^2 \equiv -1 \pmod{5}$
55	$(3 \cdot 73, 109, 54)$	No	$5 \equiv -1 \pmod{3}$
56	(223, 111, 55)	Yes	$QR(223)$ and $MH(223)$
57	(227, 113, 56)	Yes	$QR(227)$
58	$(3 \cdot 7 \cdot 11, 115, 57)$	No	$2 \equiv -1 \pmod{3}$
59	$(5 \cdot 47, 117, 58)$	No	$59 \equiv -1 \pmod{5}$
60	(239, 119, 59)	Yes	$QR(239)$
61	$(3^5, 121, 60)$	No	61 would be multiplier
62	$(13 \cdot 19, 123, 61)$	No	$2^6 \equiv -1 \pmod{13}$
63	(251, 125, 62)	Yes	$QR(251)$
64	$(3 \cdot 5 \cdot 17, 127, 63)$	Yes	$\mathbf{P}^7(\mathbf{F}_2)$ , $GMW$ and 2 new ones
65	$(7 \cdot 37, 129, 64)$	No	$5^3 \equiv -1 \pmod{7}$
66	(263, 131, 65)	Yes	$QR(263)$
67	$(3 \cdot 89, 133, 66)$	No	67 would be multiplier
68	(271, 135, 67)	Yes	$QR(271)$
69	$(5^2 \cdot 11, 137, 68)$	No	$3^2 \equiv -1 \pmod{5}$
70	$(3^2 \cdot 31, 139, 69)$	No	$2 \equiv -1 \pmod{3}$
71	(283, 141, 70)	Yes	$QR(283)$ and $MH(283)$
72	$(7 \cdot 41, 143, 71)$	No	$m = 9 \equiv 2^{55} \equiv 3^2 \pmod{287}$ would be multiplier
73	$(3 \cdot 97, 145, 72)$	No	$73^{12} \equiv -1 \pmod{97}$
74	$(5 \cdot 59, 147, 73)$	No	$2^2 \equiv -1 \pmod{5}$
75	$(13 \cdot 23, 149, 74)$	No	$m = 3^3 \equiv 5^4 \pmod{299}$ would be multiplier
76	$(3 \cdot 101, 151, 75)$	No	$m = 19 \equiv 2^9 \pmod{303}$ would be multiplier
77	(307, 153, 76)	Yes	$QR(307)$
78	(311, 155, 77)	Yes	$QR(311)$
79	$(3^2 \cdot 5 \cdot 7, 157, 78)$	No	$79 \equiv -1 \pmod{5}$
80	$(11 \cdot 29, 159, 79)$	No	$5^7 \equiv -1 \pmod{29}$
81	$(17 \cdot 19, 161, 80)$	Yes	$TP(17, 19)$
82	$(3 \cdot 109, 163, 81)$	No	$2 \equiv -1 \pmod{3}$

TABLE II (continued)

$n$	$(4n-1, 2n-1, n-1)$	exists?	examples or comment to non-existence
83	(331, 165, 82)	Yes	QR(331)
84	$(5 \cdot 67, 167, 83)$	No	$3^2 \equiv -1 \pmod{5}$
85	$(3 \cdot 113, 169, 84)$	No	$5 \equiv -1 \pmod{3}$
86	$(7^3, 171, 85)$	No	$m = 43 \equiv 2^{144} \pmod{343}$ would be multiplier
87	(347, 173, 86)	Yes	QR(347)
88	$(3^3 \cdot 13, 175, 87)$	No	$11 \equiv -1 \pmod{3}$
89	$(5 \cdot 71, 177, 88)$	No	$89 \equiv -1 \pmod{5}$
90	(359, 179, 89)	Yes	QR(359)
91	$(3 \cdot 11^2, 181, 90)$	No	$7^5 \equiv -1 \pmod{11}$
92	(367, 183, 91)	Yes	QR(367)
93	$(7 \cdot 53, 185, 92)$	No	$3^3 \equiv -1 \pmod{7}$
94	$(3 \cdot 5^3, 187, 93)$	No	$2 \equiv -1 \pmod{3}$
95	(379, 189, 94)	Yes	QR(379)
96	(383, 191, 95)	Yes	QR(383)
97	$(3^2 \cdot 43, 193, 96)$	No	97 would be multiplier
98	$(17 \cdot 23, 195, 97)$	No	$2^4 \equiv -1 \pmod{17}$
99	$(5 \cdot 79, 197, 98)$	No	$11 \equiv 3^{68} \pmod{395}$ would be multiplier
100	$(3 \cdot 7 \cdot 19, 199, 99)$	No	$4 = 2^2 \equiv 5^8 \pmod{399}$ would be multiplier

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