1. Introduction

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COMPLEX GROWTH SERIES OF COXETER SYSTEMS

by Luis PARIS [1])

1. INTRODUCTION

Let (W, S) be a *Coxeter system* (see [1] for the definitions). Throughout this paper the generating set S of W is assumed to be finite. S determines a length on W called *word length.* It is defined by

$$l(w) = l_S(w) = \min\{r \mid w = s_1 \dots s_r, s_i \in S\} ,$$

for $w \in W$. The *growth series* of W with respect to S is the formal series

$$W_S(t) = \sum_{w \in W} t^{l(w)} .$$

For a subset $X \subseteq S$, we denote by W_X the subgroup of W generated by X; the system (W_X, X) is still a Coxeter system.

With a Coxeter system (W, S) one can associate a simplicial complex $\Sigma(W, S)$, called the *Coxeter complex.* This was introduced by Tits in [5] and is an essential ingredient of the theory of buildings (see [2] and [6]).

In this paper we introduce a new formal series $W_S(t_1, t_2)$, in two variables, which will be called the *complex growth series* of (W, S), and is determined from the complex $\Sigma(W, S)$. More precisely,

$$W_S(t_1, t_2) = \sum_F t_1^{d(C_0, F)} t_2^{\operatorname{codim}(F)} ,$$

where the sum is over all the faces F of $\Sigma(W, S)$ (here we assume the empty set to be a face of $\Sigma(W, S)$ of dimension -1), and $d(C_0, F)$ is the distance between the fundamental chamber C_0 of $\Sigma(W, S)$ and the face F.

The notions of Coxeter complex, chamber, face and fundamental chamber will be recalled in Section 2.

MAIN THEOREM. *Let (W, S) be a Coxeter system. Then*

$$W_S(t_1, t_2) = \sum_{X \subseteq S} t_2^{|X|} \frac{W_S(t_1)}{W_X(t_1)} . \tag{1.1}$$

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Solomon proved in [4] that, if (W, S) is a finite Coxeter system (i.e. W is finite), then

$$(1.2) \qquad \sum_{X \subseteq S} (-1)^{|X|} \frac{W_S(t)}{W_X(t)} = t^m ,$$

where

$$m = \max_{w \in W} l(w) .$$

Later, in [1, §4.1, exercise 26], Bourbaki proved a similar formula for an infinite system (W, S); in that case,

$$(1.3) \qquad \sum_{X \subseteq S} (-1)^{|X|} \frac{W_S(t)}{W_X(t)} = 0 .$$

Several results on growth series of Coxeter groups are obtained by induction on $|S|$ using (1.2) and (1.3). We refer to [3] for an exposition on those two equalities and their applications.

An immediate corollary of (1.1), (1.2) and (1.3) is: if (W, S) is a finite Coxeter system, then

$$(1.4) \qquad W_S(t_1, -1) = t_1^m ,$$

m being the maximal length in W; and if (W, S) is an infinite Coxeter system, then

$$(1.5) \qquad W_S(t_1, -1) = 0 .$$

In fact, these two equalities (1.4) and (1.5) can and will be proved independently of the formulas (1.1), (1.2) and (1.3) (Proposition 2).

As an illustration of the Main Theorem, let us give two explicit examples.

1) Assume

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 \rangle$$

to be a Coxeter group of type A_2. The geometric realisation of $\Sigma(W, S)$ is an hexagon. We have

$$W_S(t_1, t_2) = (1 + t_1)(1 + t_1 + t_1^2) + 2(1 + t_1 + t_1^2)t_2 + t_2^2 ,$$

and

$$W_{\{s_1, s_2\}}(t) = (1 + t)(1 + t + t^2) ,$$
$$W_{\{s_1\}}(t) = W_{\{s_2\}}(t) = 1 + t,$$
$$W_\emptyset(t) = 1.$$

Thus the equality (1.1) holds in that case.

2) Assume

$$W = < s_1 , s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_1 s_3)^3 = (s_2 s_3)^3 = 1 >$$

to be a Coxeter group of type $\tilde{A}_2$. The geometric realisation of $\Sigma(W, S)$ is a plane. We have

$$W_S(t_1, t_2) = \frac{(1+t_1)(1+t_1+t_1^2)}{(1-t_1)(1-t_1^2)} + 3\frac{1+t_1+t_1^2}{(1-t_1)(1-t_1^2)} t_2$$
$$+ 3\frac{1}{(1-t_1)(1-t_1^2)} t_2^2 + t_2^3 ,$$

and

$$W_S(t) = \frac{(1+t)(1+t+t^2)}{(1-t)(1-t^2)} ,$$
$$W_{\{s_1, s_2\}}(t) = W_{\{s_1, s_3\}}(t) = W_{\{s_2, s_3\}}(t) = (1+t)(1+t+t^2),$$
$$W_{\{s_1\}}(t) = W_{\{s_2\}}(t) = W_{\{s_3\}}(t) = 1 + t,$$
$$W_\emptyset(t) = 1.$$

Thus the equality (1.1) holds in that case.

In Section 2 we will recall some definitions in the theory of Coxeter complexes, we will define the complex growth series of a Coxeter system (W, S), we will prove that $W_S(t_1, 0) = W_S(t_1)$ and $W_S(0, t_2) = (1+t_2)^{|S|}$ (Proposition 1), we will prove the equalities (1.4) and (1.5) (Proposition 2), and we will prove the Main Theorem.

2. Complex growth series

We assume the reader to be familiar with the notions of simplicial complex, chamber complex, adjacency between two chambers, gallery and labelling. We refer to [2, Chap. I, Appendix] for a good exposition of these notions.

Let (W, S) be a Coxeter system. A *special coset* of (W, S) is a coset wW_X, with $w \in W$ and $X \subseteq S$. We denote by $\Sigma = \Sigma(W, S)$ the poset of all special cosets, ordered by the reverse inclusion; $B \leqslant A$ in Σ if $B \supseteq A$ in W. The poset Σ is a labelled chamber simplicial complex (see [2, Chap. III, § 1]).

A *chamber* of Σ is a singleton $\{w\}$ with $w \in W$. A *vertex* of Σ is a special coset $wW_{S-\{s\}}$ with $w \in W$ and $s \in S$. The face of Σ of dimension -1 is the