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Moreover, even these important works in algebra were viewed in the 19th century, in the overall mathematical scheme, as secondary. The primary mathematical fields in that century were analysis (complex analysis, differential equations, real analysis), and geometry (projective, noneuclidean, differential, and algebraic). But after the work of Emmy Noether and others in the 1920s, algebra became central in mathematics.

It should be noted that Emmy Noether was not the only, nor even the only major, contributor to the abstract, axiomatic approach in algebra. Among her predecessors who contributed to the genre were Cayley and Frobenius in group theory, Dedekind in lattice theory, Weber and Steinitz in field theory, and Wedderburn and Dickson in the theory of hypercomplex systems. Among her contemporaries, Albert in the U.S. and Artin in Germany stand out.

The "big bang" theory rarely applies when dealing with the origin of mathematical ideas. So also in Emmy Noether's case. The concepts she introduced and the results she established must be viewed against the background of late-19th-and early-20th-century contributions to algebra. She was particularly influenced by the works of Dedekind. In discussing her contributions she frequently used to say, with characteristic modesty: "It can already be found in Dedekind's work" ("Es steht schon bei Dedekind") ([12], p. 68). In commenting on them, I will thus be considering their roots in Dedekind's work and in that of others from which she drew inspiration and on which she built.

Emmy Noether contributed to the following major areas of algebra: invariant theory (1907-1919), commutative algebra (1920-1929), noncommutative algebra and representation theory (1927-1933), and applications of noncommutative algebra to problems in commutative algebra (1932-1935). She thus dealt with just about the whole range of subject-matter of the algebraic tradition of the 19th and early 20th centuries (with the possible exception of group theory proper). What is significant is that she transformed that subject-matter, thereby originating a new algebraic tradition — what has come to be known as modern or abstract algebra.

I will now discuss Emmy Noether's contributions to each of the above areas.

INVARIANT THEORY

Emmy Noether's statement (quoted above), that her ideas are already in Dedekind's work, could, with equal validity, have been put as "It all started with Gauss". Indeed, invariant theory dates back to Gauss' study of binary quadratic forms in his *Disquisitiones Arithmeticae* of 1801. Gauss defined an equivalence relation on such forms and showed that the discriminant is an invariant of the form under equivalence (see [1]). A second important source of invariant theory is projective geometry, which originated in the 1820s. A significant problem was to distinguish euclidean from projective properties of geometric figures. The projective properties turned out to be those invariant under "projective transformations" (see [26], [31]).

Formally, invariant theory began with Cayley and Sylvester in the late 1840s. Cayley used it to bring to light the deeper connections between metric and projective geometry (see [10]). Although important connections with geometry were maintained throughout the 19th and early 20th centuries, invariant theory soon became an area of investigation independent of its relations to geometry. In fact, it became an important branch of *algebra* in the second half of the 19th century. To Sylvester "all algebraic inquiries, sooner or later, end at the Capitol of modern algebra over whose shining portal is inscribed the Theory of Invariants" ([26], p. 930).

An important problem of the abstract theory of invariants was to discover invariants of various "forms".[1]) Many of the major mathematicians of the second half of the 19th century worked on the computation of invariants of specific forms. This led to the major problem of invariant theory, namely to determine a complete system of invariants (a basis) for a given form; i.e., to find invariants of the form — it was conjectured that finitely many would do — such that every other invariant could be expressed as a combination of these. Cayley showed in 1856 that the finitely many invariants he had found earlier for binary quartic forms (i.e., forms of degree four in two variables) are a complete system. About ten years later Gordan proved that every binary form (of any degree) has a finite basis. Gordan's proof of this important result was computational – he *exhibited* a complete system of invariants.[2]) In 1888 Hilbert astonished the mathematical world by announcing a new, conceptual, approach to the problem of invariants. The idea was to consider, instead of invariants, expressions in a finite number of variables, in short, the polynomial ring in those variables. Hilbert then proved what came to be

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[1]) E.g., a *binary form* is an expression of the form $f(x_1, x_2) = a_0x_1^n + a_1x_1^{n-1}x_2 + \dots + a_nx_2^n$. If this form is transformed by a linear transformation T of the variables x_1 and x_2 into the form $F(X_1, X_2) = A_0X_1^n + A_1X_1^{n-1}X_2 + \dots + A_nX_2^n$, then any function I of the coefficients of f which satisfies the relation $I(A_0, \dots, A_n) = r^kI(a_0, \dots, a_n)$ is called an *invariant* of f under T (r denotes the determinant of T).

[2]) Weyl observed that "there exist papers of his [Gordan's] where twenty pages of formulas are not interrupted by a single word; it is told that in all his papers he himself wrote the formulas only, the text being added by his friends" ([41], p. 117).

known as the Basis Theorem, namely that every ideal in the ring of polynomials in finitely many variables has a finite basis. A corollary was that every form (of any degree, in any number of variables) has a finite complete system of invariants. Gordan's reaction to Hilbert's proof, which did not explicitly exhibit the complete system of invariants, was that "this is not mathematics; it is theology" ([26], p. 930).[1])

Emmy Noether's thesis, written under Gordan in 1907, was entitled "On Complete Systems of Invariants for Ternary Biquadratic Forms". The thesis was computational, in the style of Gordan's work. It ended with a table of the complete system of 331 invariants for such a form. Noether was later to describe her thesis as "a jungle of formulas" ([24], p. 11).[2])

Emmy Noether obtained, however, several notable results on invariants during the 1910s. First, using the methods she had developed in two papers (in 1915 and 1916) on the subject, she made a significant contribution to the problem, first posed by Dedekind, of finding a Galois extension of a given number field with a prescribed Galois group.[3]) Second, during her work in Göttingen on differential invariants, she used the calculus of variations to obtain the so-called Noether Theorem, still important in mathematical physics (see [7], p. 125). The physicist Fez Gursey says of this contribution ([22], p. 23):

> The key to the relation of symmetry laws to conservation laws in physics is Emmy Noether's celebrated theorem which states that a dynamical system described by an action under a Lie group with n parameters admits n invariants (conserved quantities) that remain constant in time during the evolution of the system.

Alexandrov summarizes her work on invariants by noting that it "would have been enough... to earn her the reputation of a first class mathematician" ([2], p. 156).

What was the route that led Emmy Noether from the computational theory of invariants to the abstract theory of rings and modules?[4]) In 1910 Gordan retired from the University of Erlangen and was soon replaced by Ernst

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[1]) Later Hilbert gave a constructive proof of his result which, however, he did not consider significant, but which elicited from Gordan the statement: "I have convinced myself that theology also has its advantages" ([26], p. 930).

[2]) When asked in 1932 to review a paper on invariants, she refused, declaring "I have completly forgotten all of the symbolic calculations I ever learned" ([12], p. 18).

[3]) The problem, in this generality, is still unresolved, although it has been solved for symmetric and solvable groups (see [7], p. 115).

[4]) "A greater contrast is hardly imaginable than between her first paper, the dissertation, and her works of maturity", remarks Weyl ([41], p. 120).

Fischer. He, too, was a specialist in invariant theory, but invariant theory of the Hilbert persuasion. Emmy Noether came under his influence and gradually made the change from Gordan's algorithmic approach to invariant theory to Hilbert's conceptual approach. Later work on invariants brought her in contact with the famous joint paper of Dedekind and Weber (see p. 115 below) on the arithmetic theory of algebraic functions. She became "sold" on Dedekind's approach and ideas, and this determined the direction of her future work.

COMMUTATIVE ALGEBRA

The two major sources of commutative algebra are algebraic geometry and algebraic number theory. Emmy Noether's two seminal papers of 1921 and 1927 on the subject can be traced, respectively, to these two sources. In these papers, entitled, respectively, Ideal Theory in Rings (Idealtheorie in Ringbereichen) and Abstract Development of Ideal Theory in Algebraic Number Fields and Function Fields (Abstrakter Aufbau der Idealtheorie in algebraischen Zahl-und Funktionenkörpern), she broke fundamentally new ground, originating "a new and epoch-making style of thinking in algebra" ([41], p. 130).

Algebraic geometry had its origins in the study, begun in the early 19th century, of abelian functions and their integrals. This analytic approach to the subject gradually gave way to geometric, algebraic, and arithmetic means of attack. In the algebraic context, the main object of study is the ring of polynomials $k[x_1, x_2, ..., x_n]$, k a field (in the 19th century k was the field of real or complex numbers). Hilbert in the 19th century, and Lasker and Macauley in the early 20th century, had shown that in such a ring every ideal is a finite intersection of primary ideals, with certain uniqueness properties.[1]) (Geometrically, the result says that every variety is a unique, finite, union of irreducible varieties.) In her 1921 paper Emmy Noether generalized this result to arbitrary commutative rings with the ascending chain condition (a.c.c.).[2]) Her main result was that in such a ring every ideal is a finite intersection (with accompanying uniqueness properties) of primary ideals. (See [14] for historical and [3] for technical details.)

What was so significant about this paper which (we recall) MacLane singled out as marking the beginning of abstract algebra as a conscious discipline?

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[1]) An ideal I in a commutative ring R is called *primary* if $xy \in I$ implies $x \in I$ or $y^t \in I$ for some positive integer t. The concept of primary ideal is an extension to rings of prime power for the integers.

[2]) A commutative ring R satisfies the *ascending chain condition* if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ terminates; i.e., $I_n = I_{n+1} = \ldots$ for some positive integer n.