Noncommutative algebra and representation theory

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satisfies both the a.c.c. and d.c.c., (c) that if an R-module M is finitely generated and R satisfies the a.c.c. (d.c.c.), then so does M.

To summarize Emmy Noether's contributions to commutative algebra: in addition to proving important results, she introduced concepts and developed techniques which have become standard tools of the subject. In fact, her 1921 and 1927 papers, combined with those of Krull of the 1920s, are said to have created the subject of commutative algebra.

NONCOMMUTATIVE ALGEBRA AND REPRESENTATION THEORY

Before her ideas in commutative algebra had been fully assimilated by her contemporaries, Emmy Noether turned her attention to the other major algebraic subjects of the 19th and early 20th centuries, namely hypercomplex number systems (what we now call associative algebras) and groups (in particular, group representations). She extended and unified these two subjects through her abstract, conceptual approach, in which module-theoretic ideas that she had used in the commutative case played a crucial role.

The theory of hypercomplex systems began with Hamilton's 1843 introduction of the quaternions. At the end of the 19th century, E. Cartan, Frobenius, and Molien gave structure theorems for such systems over the real and complex numbers, and in 1907 Wedderburn extended these to hypercomplex systems over arbitrary fields. In the spirit of Emmy Noether's work in commutative algebra, Artin extended Wedderburn's results to (noncommutative, semisimple) rings with the descending chain condition. (See [25] for details.)

Groups were the first algebraic systems to be developed extensively. By the end of the 19th century they began to be studied abstractly. An important tool in that study was representation theory, developed by Burnside, Frobenius, and Molien in the 1890s (see [20]). The idea was to study, instead of the abstract group, its concrete representations in terms of matrices (A *representation* of a group is a homomorphism of the group into the group of invertible matrices of some given order.)

In her 1929 paper Hypercomplex Numbers and Representation Theory (Hyperkomplexe Grössen und Darstellungstheorie) Emmy Noether framed group representation theory in terms of the structure theory of hypercomplex systems. The main tool in this approach was the *module.* The idea was to associate with each representation ϕ of G by invertible matrices with entries in some field k, a $k(G)$-module V called the *representation module* of ϕ($k(G)$ is the *group algebra* of G over k). Conversely, any $k(G)$-module M gives rise to a representation ψ of G.[1]) This establishes a one-one correspondence between representations of G (over k) and $k(G)$-modules. The standard concepts of representation theory can now be phrased in terms of modules. For example, two representations are equivalent if and only if their representation modules are isomorphic; a representation is irreducible if and only if its representation module is simple. The techniques of module theory, and the structure theory of hypercomplex systems (applied to the hypercomplex system $k(G)$) can now be used to "recast the foundations of group representation theory" ([27], p. 150). (See [27] for historical and [11] for technical details.)

Noether's work in this area created a very effective conceptual framework in which to study representation theory. For example, while the (computational) classical approach to representation theory is valid only over the field of complex numbers (or, at best, over an algebraically closed field of characteristic 0), Noether's approach remains meaningful for any field (of any characteristic). The use of general fields in representation theory became important in the 1930s when Brauer began his pioneering studies of *modular representations* (i.e., those in which the characteristic of the field divides the order of the group). Noether's ideas also "planted the seed of modern integral representation theory" ([27], p. 152), that is, representation theory over commutative rings rather than over fields. Noether herself extended the representation theory of groups to that of semi-simple artinian rings; here she needed the concept of a *bimodule.*

A word about modules, which were so central in Emmy Noether's work in both commutative and noncommutative algebra. Dedekind, in connection with his 1871 work in algebraic number theory, was the first to use the term "module", but to him it meant a subgroup of the additive group of complex numbers (i.e., a Z-module); in 1894 he developed an extensive theory of such modules. Lasker, in his 1905 work on decomposition of polynomial rings, used the terms "module" and "ideal" interchangeably (the former he applied to polynomial rings over **C**, the latter to such rings over **Z**). Noether was the first to use the notion of module abstractly (with a ring as domain of operators) and to recognize its potential. In fact, it is through her work that the concept of module became the central concept of algebra that it is today. Indeed, modules are important not only because of their unifying, but also because of their *linearizing,* power. (They are, after all, generalizations of vector

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[1]) In one direction, consider ϕ as a homomorphism of G into $L(V, V)$, the set of linear transformations of a vector space V over k. We turn V into a $k(G)$-module by defining $v \cdot g = \phi(g)(v)$, for $v \in V$, $g \in G$, and extending by linearity to all of $k(G)$. In the other direction, define $\psi: G \to L(M, M)$ by $\psi(g)(m) = m \cdot g$. See [11], Chapter II, for details.

spaces, and many of the standard vector-space constructions, such as subspace, quotient space, direct sum, and tensor product carry over to modules.)[1]) In fact, the importance of the invention of *homological algebra* was that it carried the process of linearization far forward by developing tools for its implementation. (E.g., the functors "Ext" and "Tor" measure the extent to which modules over general rings "misbehave" when compared to modules over fields, viz. vector spaces; see [8].)

APPLICATIONS OF NONCOMMUTATIVE TO COMMUTATIVE ALGEBRA

Noether believed that the theory of noncommutative algebras is governed by simpler laws than that of commutative algebra. In her 1932 plenary address at the International Congress of Mathematicians in Zurich, entitled Hypercomplex Systems and their Relations to Commutative Algebra and Number Theory (Hyperkomplexe Systeme in ihren Beziehungen zur kommutativen Algebra und Zahlentheorie), she outlined a program putting that belief into practice. Her program has been called "a foreshadowing of modern cohomology theory" ([35], p. 8). The ideas on factor sets contained therein were soon used by Hasse and Chevalley "to obtain some of the main results on global and local class field theory" ([22], p. 26). Noether's own immediate objective was to apply the theory of central simple algebras (as developed by her, Brauer, and others) to problems in class field theory. (See [7], [35], and [36].)

Some of her ideas (and those of others) on the interplay between commutative and noncommutative algebra had already recently born fruit with the proof of the celebrated Albert-Brauer-Hasse-Noether Theorem. This result, called by Jacobson "one of the high points of the theory of algebras" ([22], p. 21), gives a complete description of finite-dimensional division algebras over algebraic number fields.[2]) It is important in the study of finite-dimensional algebras and of group representations.

To bring out the context of the above theorem, it should be noted that Wedderburn's 1907 structure theorems for finite-dimensional algebras reduced their study to that of nilpotent algebras and division algebras. Since the unravelling of the structure of the former seemed (and still seems, despite considerable progress) "hopeless", attention focussed on the latter.

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[1]) We know the power of linearization in analysis. Modules can be said to provide analogous power in algebra.

[2]) They are intimately related to the "cyclic" algebras studied earlier by Dickson (see [21], Vol. II, p. 480).