AN ANALOGUE OF HUBER'S FORMULA FOR RIEMANN'S ZETA FUNCTION

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AN ANALOGUE OF HUBER'S FORMULA FOR RIEMANN'S ZETA FUNCTION

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To the memory of Michio Kuga

1. INTRODUCTION

A remarkable formula of H. Huber [11] relates the class 1 spectrum of a compact Riemann surface X and the spectral zeros of Selberg's zeta function ζ_X of X. More generally, if X is a space form (not necessarily compact) of a rank 1 symmetric space one can still assign to X a Selberg zeta function ζ_X and formulate a generalized version of Huber's formula [6], [18]. Here a decisive role is played by the Selberg trace formula.

On the other hand Weil's explicit sum formula for Riemann's zeta function ζ [16], [17] bears some striking similarity in appearance to the trace formula. It is now known, as a matter of fact, by a recent work of D. Goldfeld [7] that there exists a kernel function on a suitable space such that the conjugacy class sum in Selberg's trace formula is precisely the sum over the primes of Weil's formula; i.e. Weil's formula indeed can be interpreted as a trace formula.

Motivated by a certain "radial" function which occurs in semisimple Lie theory we consider a certain test function which we plug into Weil's formula, and we derive thereby a formula of Huber type for ζ. The formula, see Theorem 7.1, involves a sum over the "spectrum" of ζ — i.e. over its non-trivial zeros. We derive a second interesting formula in Theorem 7.10 by specializing the parameter s in Theorem 7.1.

Although Lie theory and spectral theory serve as a context and motivation, the lecture which is largely self-contained requires no familiarity with these subjects. We assume only a knowledge of basic real and complex analysis.

[1]) This is an expanded version of an invited Mathematical Association of America address delivered at the winter San Francisco meeting on January 16, 1991.

The author is thankful for and honored by the invitation extended by the Program Committee of the Mathematical Association of America to deliver this lecture. We dedicate the lecture to the memory of a very great mathematician — a kind and humble man — a friend — Professor Michio Kuga.

2. Outline of the Lecture

I. Huber's formula (as a context)
II. Riemann's zeta function — basic facts and the Riemann hypothesis (= RH)
III. Test functions
IV. Weil's explicit formula
V. The Schwartz space and the RH
VI. The main test function
VII. An analogue of Huber's formula

I. Huber's formula (as a context)

Since Huber's formula provides the motivation for this lecture we shall state (for the record) this remarkable result. Neither the result nor any understanding of it is required for later purposes.

Let G denote the group $SL(2, \mathbf{R})$ of real 2×2 matrices with determinant equal to 1:

$$G = SL(2, \mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{matrix} a, b, c, d, \in \mathbf{R} \\ ad - cb = 1 \end{matrix} \right\} \tag{1.1}$$

where $\mathbf{R}$ denotes the field of real numbers. Let $\Gamma \subset G$ be a discrete torsionfree[1]) subgroup such that the quotient $\Gamma \backslash G$ is compact. Euler's classical gamma function will also be denoted by Γ:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt\,, \qquad \operatorname{Re} s > 0\,. \tag{1.2}$$

The function $N: \Gamma - \{1\} \to \mathbf{R}$ defined by

$$N(P) = \max |c|^2\,, \qquad c = \text{an eigenvalue of } P \tag{1.3}$$

[1]) That is, if $\gamma^n = 1$ for $\gamma \in \Gamma, n > 0$ an integer, then $\gamma = 1$.

for $P \in \Gamma - \{1\}$ is called the *norm* of Γ. We shall particularly be interested in the restriction of N to the *prime* or primitive elements P of Γ; i.e. elements P which generate their centralizer in Γ: $Z_\Gamma(P) = \{P\}$ = a cyclic group. Given the norm function N we have Selberg's zeta function ζ [15], [5], [10] given by

$$(1.4)\qquad \zeta(s) = \prod_{P = \text{a prime}} \prod_{k=0}^{\infty} [1 - N(P)^{-s-k}]\,, \qquad \operatorname{Re} s > 1\,.$$

ζ, which admits a full analytic continuation in the complex plane, has a series of "topological" zeros (= "trivial" zeros) and certain non-trivial "spectral" zeros $s_j^{\pm} = \frac{1}{2} \pm i\left(\lambda_j - \frac{1}{4}\right)$ where the λ_j (with $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$) are the eigenvalues of the Laplacian $-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on the upper half-plane $\prod^+$ projected to $\Gamma\backslash\prod^+$. Here, recall that G acts transitively on $\prod^+$ by linear fractional transformations:

$$(1.5)\qquad g.z. \overset{\text{def}}{=} \frac{az+b}{cz+d} \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G\,, \qquad \operatorname{Im} z > 0\,.$$

Moreover if $SO(2)$ is the compact subgroup of orthogonal matrices in G then (by transitivity of the G-action) $\prod^+ = G/SO(2)$, and $X = X_\Gamma \overset{\text{def}}{=} \Gamma\backslash\prod^+ = \Gamma\backslash SL(2, \mathbf{R})/SO(2)$ is the typical compact Riemann surface of genus $\geqslant 2$; Γ is the fundamental group of X. We may (and should) denote ζ in (1.4) by ζ_X, as in the introduction.

If $n_j \geqslant 0$ is the (finite) multiplicity of λ and $\operatorname{vol}(\Gamma\backslash G) \overset{\text{def}}{=} \int_{\Gamma\backslash G} 1\, dx$ where dx is a G-invariant measure on $\Gamma\backslash G$ suitably normalized[1]), then in terms of the above definitions and notation one has the following remarkable formula of Huber [11]: For $s \in \mathbf{C}$ (the field of complex numbers) with $\operatorname{Re} s > 1$

$$(1.6)\qquad \frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \sum_{j=0}^{\infty} n_j \Gamma\left(\frac{s-s_j^+}{2}\right)\Gamma\left(\frac{s-s_j^-}{2}\right) = \operatorname{vol}(\Gamma\backslash G)$$

$$+ \frac{\pi \cdot 2^{-s+3/2}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \sum_{\substack{\text{certain}\\ \text{primes}\\ p \in \Gamma - \{1\}}} \sum_{j=1}^{\infty} \frac{\log N(p)}{N(p)^{j/2} - N(p)^{-j/2}} [\cosh j \log N(p)]^{-s+1/2}\,.$$

[1]) dx is unique up to a positive constant.

In case $\Gamma \backslash G$ is non-compact, but $\mathrm{vol}(\Gamma \backslash G) < \infty$, the zeta function ζ_X is still defined and (1.6) remains valid provided the n_j are interpreted as the multiplicity of eigenvalues of the discrete spectrum of X_Γ, and provided extra terms are added to account for contributions via the "continuous" spectrum of X_Γ. Such terms, for example, may have the form $\int_R \hat{f}_1 \frac{\Gamma'}{\Gamma} dx$ or $\int_R \hat{f}$ [trace of the logarithmic derivative of the "scattering matrix" of an Eisenstein series] dx where dx also denotes Lebesgue measure on $\mathbf{R}$ and $\hat{f}$ denotes the Fourier transform of a function f on $\mathbf{R}$. In this lecture we normalize the definition of $\hat{f}$, say $f \in L^1(\mathbf{R})$, by

$$(1.7) \qquad \hat{f}(y) = \int_{\mathbf{R}} e^{iyx} f(x)\, dx \quad \text{for} \quad y \in \mathbf{R}\,.$$

Then the Fourier inversion formula is

$$(1.8) \qquad f(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-ixy} \hat{f}(y)\, dy$$

for almost all $x \in \mathbf{R}$, for $\hat{f} \in L^1(\mathbf{R})$. In particular (1.8) holds for all $x \in \mathbf{R}$ if f is continuous.

In contrast to Huber's original proof, (1.6) follows by plugging a certain test function into Selberg's trace formula and a computation of the "spherical" Fourier transform of that test function [14], [6], [18]. Moreover, the n_j in (1.6) can be shown to coincide with the multiplicity of the spectral zeros $s_j^{\pm}$, say $s_j^+ \neq s_j^-$ [5] [1]). Therefore the following diagram captures some of the features of formula (1.6) (roughly).

[1]) If $s_j^+ = s_j^-$ then s_j^+ has order $2n_j$.

Huber's formula (1959)

⇑ specific Fourier transform computation

Selberg trace formula (1956)

basic Lie theory

features:

(i) non-trivial zeros $s_j^{\pm}$ of Selberg's zeta function (1956) and their multiplicity n_j; more precisely a sum of the form

$$\sum_j n_j \Gamma\left(\frac{s-s_j^+}{2}\right)\Gamma\left(\frac{s-s_j^-}{2}\right)$$

for a fixed parameter s, $\operatorname{Re} s > 1$

(ii) a discrete sum involving the log of the norm of certain prime elements

(iii)

$$\int_{\mathbf{R}} \begin{array}{c}\text{certain}\\ \text{Fourier}\\ \text{transform}\end{array} \cdot \begin{array}{c}\text{logarithmic}\\ \text{derivative of}\\ \text{the gamma}\\ \text{function}\end{array} \, dx$$

(in case $\Gamma \backslash G$ is non-compact)

On the other hand within the context of basic number theory [1]) one has (applying the "analogy functor") the following parallel of the preceding diagram (in the form of a question)

what corresponding formula?

⇑ what Fourier transform computation?

Weil's formula (1952) (cf. remarks of the introduction)

[1]) All of the basic facts we need will be presented in the next section.

(i) non-trivial zeros of Riemann's zeta function (1859) and their multiplicity

(ii) the classical von Mangoldt function

(iii) as before

The purpose of the lecture is to present the "corresponding formula" (i.e. the analogue of Huber's formula in the context of elementary number theory). Thus we shall introduce a specific test function, whose Fourier transform can be determined, and apply Weil's formula, as indicated in the introduction.

As a closing remark for this section we note (for the record) that one can indeed assign a "von Mangoldt function" Λ to the pair (G, Γ). Namely, every $\gamma \in \Gamma - \{1\}$ is the power of a unique prime $p: \gamma = p^{j(\gamma)}$ for some unique integer $j(\gamma) \geqslant 1$. One sets

(1.9) $$\Lambda(\gamma) \overset{\text{def}}{=} \frac{\log N(p)}{1 - N(\gamma)^{-1}} .$$

II. Riemann's zeta function - basic facts and the Riemann hypothesis (= RH)

Riemann's zeta function ζ is defined by

(2.1) $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for} \quad \operatorname{Re} s > 1 .$$

This function was also considered by Euler (100 years before Riemann). The basic facts concerning ζ are summarized as follows:

THEOREM 2.2. (i) ζ *is holomorphic on* $\operatorname{Re} s > 1$ (since the series in (2.1) converges uniformly on compact subsets of $\operatorname{Re} s > 1$) *and* ζ *extends to a meromorphic function* ζ *on* $\mathbf{C}$ *having exactly one pole:* $s = 1$ *is simple with residue* $= 1$

(ii) ζ *satisfies a functional equation*

$$s \to 1 - s : \frac{\pi^{s/2}\zeta(1-s)}{\Gamma\left(\frac{s}{2}\right)} = \frac{\pi^{\frac{1-s}{2}}\zeta(s)}{\Gamma\left(\frac{1-s}{2}\right)} .$$

(iii) ζ *has an Euler product representation:*

(2.3) $$\zeta(s) = \prod_{p = \text{prime} > 0} \frac{1}{1 - p^{-s}} \quad \text{for} \quad \operatorname{Re} s > 1 .$$

Note that equation (1.4) for Selberg's function is the analogue of (2.3).

(iv) $\zeta(s) = 0$ *for* $s = -2, -4, -6, -8\ldots$ This follows from (ii) since $\frac{1}{\Gamma(s)} = 0$ for $s = -1, -2, -3, -4, \ldots$ $\zeta(0) \neq 0$ as in fact $\zeta(0) = -\frac{1}{2}$. *The zeros* $\{-2n\}_{n=1}^{\infty}$ *are called the trivial zeros of* ζ.

(v) *If* $\zeta(s) = 0$ *and* s is non-trivial (i.e. $s \neq -2n$ *for some* $n = 1, 2, 3, 4, \ldots$*)* *then* $s \in \mathbf{C} - \mathbf{R}$ and $0 < \operatorname{Re} s < 1$. *The world famous Riemann Hypothesis* (RH) *(which remains un-proved) states that for such an* s, $\operatorname{Re} s = \frac{1}{2}$!

(vi) $\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ *where* Λ *is the (von Mangoldt) function defined by*

$$\Lambda(n) \stackrel{\text{def}}{=} \begin{cases} \log p & \text{if} \quad n = p^k\,,\ p = \text{prime}\,, \\ & \qquad k, p > 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.4}$$

Some remarks. Definition (1.9) is the analogue of definition (2.4). $\frac{\zeta'}{\zeta}(0) = \log 2\pi$.

Let $\psi(x) = \sum_{1 \leqslant n \leqslant x} \Lambda(n)$ for $x > 1$. This function $\psi: x \to \psi(x)$ (Chebyshev's function) is the subject of the celebrated *prime number theorem* (PNT) which states that $\psi(x) \sim x$ as $x \to \infty$; i.e. $\lim_{x \to 1} \frac{\psi(x)}{x} = 1$. Equivalently, if π is the function which counts the number of primes not exceeding a given number ($\pi(x) = \sum_{1 \leqslant p \leqslant x,\, p = a \ \text{prime}}$) then the PNT gives the asymptotic growth of π at ∞: $\pi(x) \sim \frac{x}{\log x}$: i.e. $\lim_{x \to \infty} \pi(x) \frac{\log x}{x} = 1$. This celebrated result has an interesting history going back to Legendre, Gauss, Chebyshev, Hadamard, de la Vallée Poussin, and others; cf. [12].

III. Test functions

Suppose g is a measurable function on $\mathbf{R}$ which satisfies $|g(x)| \leqslant Me^{-b|x|}\ \forall x \in \mathbf{R}$, for some $M, b > 0$. Then the function $x \to e^{isx}g(x)$ on $\mathbf{R}$ is in $L^1(\mathbf{R})$ for $|\operatorname{Im} s| < b$. One can thus define the *complex* Fourier transform $\hat{g}$ of g by

(3.1) $$\hat{g}(s) = \int_{\mathbf{R}} e^{isx} g(x)\,dx$$

for $-b < \operatorname{Im} s < b$; cf. (1.7). $\hat{g}$ is holomorphic on the strip $-b < \operatorname{Im} s < b$.

Although we could consider a broader class of functions (as considered in chapter 17 of [13], or even more generally in [1]) the following definition will suffice for our purpose.

Definition 3.2. A *test function* is a continuously differentiable function g on $\mathbf{R}$ which satisfies

(3.3) $$|g(x)| \leqslant Me^{-b|x|}, \quad |g'(x)| \leqslant M_1 e^{-b|x|}$$

$\forall x \in \mathbf{R}$, where $M, M_1 > 0$ and $b > \frac{1}{2}$; g' = the derivative of g.

In application the function g which we consider later will in fact be infinitely differentiable. It is easy to check that a continuously differentiable function with compact support is a test function (where one takes $b = 1 > 1/2$).

We shall need the "shifted" Fourier transform g^* of a measurable function g satisfying $|g(x)| \leqslant Me^{-b|x|}$:

(3.4) $$g^*(s) \overset{\text{def}}{=} \hat{g}\left(i\left(\frac{1}{2} - s\right)\right) = \int_{\mathbf{R}} e^{\left(s-\frac{1}{2}\right)x} g(x)\,dx\,.$$

By the above remarks g^* is defined and holomorphic on $\frac{1}{2} - b < \operatorname{Re} s < b + \frac{1}{2}$.

IV. Weil's explicit formula

We turn now to Weil's formula mentioned in the introduction. The formula has been formulated, quite generally, for so-called L-functions attached to Hecke grösencharacters in the context of an algebraic number field K [16], [17]. We shall consider Weil's formula only in regard to the Riemann zeta function ζ: K = the field of rational numbers. The reader with interests in the general formula for arbitrary K may consult [13], [1] for detailed proofs.

The typical non-trivial zero of ζ will be denoted by p and we shall write [1]).

(4.1) $$n_p = \text{the multiplicity of } p\,.$$

[1]) See Theorem 2.2, parts (iv) (v).

The logarithmic derivative of Γ (see (1.2)) will always be denoted by ψ:

$$\psi \overset{\text{def}}{=} \Gamma'/\Gamma . \tag{4.2}$$

ψ is a meromorphic function whose poles (all of which are simple) are $0, -1, -2, -3, -4, \ldots$; the residue at each pole is -1.

THEOREM 4.3 (Weil's Explicit Formula). *Let g be a test function (Definition 3.2) with shifted Fourier transform g^* (definition (3.4)) and let Λ be the von Mangoldt function (definition (2.4)). Then in the notation of (3.1), (4.1), and (4.2)*

$$\begin{aligned}\lim_{T\to\infty} \sum_{\substack{p \\ |\operatorname{Im} p| < T}} n_p\, g^*(p) \left(\overset{\text{def}}{=} \sum\nolimits_p g^*(p)\right) &= \hat{g}\left(\frac{i}{2}\right) + \hat{g}\left(-\frac{i}{2}\right) - g(0)\log\pi \\ &- \sum\nolimits_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\,[g(\log n) + g(-\log n)] + \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T} \hat{g}(t)\operatorname{Re}\psi\left(\frac{1}{4} + i\frac{t}{2}\right)dt\end{aligned} \tag{4.4}$$

where all limits here are finite

Summation formulas quite similar in spirit to (4.4) are given in [3], [9]. The prototype of such formulas is the explicit formula of von Mangoldt in the theory of prime numbers:

$$\psi(x) = x - \lim_{T\to\infty} \sum_{|\operatorname{Im} p| < T} n_p \frac{x^p}{p} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \tag{4.5}$$

for $x > 1$ (say x non-integral), for ψ the Chebyshev function defined following (2.4); cf. Theorem 29 of [12].

V. THE SCHWARTZ SPACE AND THE RH

Let $C^\infty(\mathbf{R})$ be the space of infinitely differentiable functions on $\mathbf{R}$ and let $C_c^\infty(\mathbf{R})$ be the subspace of functions in $C^\infty(\mathbf{R})$ which have compact support. For $f \in C^\infty(\mathbf{R})$ and integers $n, m \geqslant 0$ let

$$p_{n,m}(f) = \sup \left| x^n \frac{d^m f}{dx^m}(x) \right| \quad \text{over} \quad x \in \mathbf{R} . \tag{5.1}$$

The *Schwartz Space* $S(\mathbf{R})$ of $\mathbf{R}$ is the vector space $\{f \in C^\infty(\mathbf{R}) \mid p_{n,m}(f) < \infty\ \forall n, m\}$. Equation (5.1) defines a family $\{p_{n,m}\} n, m \geqslant 0$ of seminorms

$p_{n,m}$ on $S(\mathbf{R})$, which therefore generate a locally convex topology τ on $S(\mathbf{R})$[1]). τ is described as follows. For $\varepsilon > 0$, F a finite collection of the integers n, m, and $f \in S(\mathbf{R})$, let

$$N(f, \varepsilon, F) = \{f_1 \in S(\mathbf{R}) \mid p_{n,m}(f_1 - f) < \varepsilon \forall n, m \in F\} .$$

Then for $u \subset S(\mathbf{R})$, u is *open* $\leftrightarrow \forall f \in u \,\exists$ some (ε, F) such that $N(f, \varepsilon, F) \subset u$; i.e. $\{N(f, \varepsilon, F)\}$ is a basis of τ.

The canonical example of a Schwartz function (i.e. a function in $(S(\mathbf{R}))$ is the function $x \to e^{-ax^2}$, where $a > 0$ is fixed. Another example (which is the key example for this lecture) is the function $g_z : x \to (\cosh x)^{-z}$ where $z \in \mathbf{C}$ is fixed, $\mathrm{Re}\, z > 0$. To see that g_z is indeed a Schwartz function one checks by induction the following.

PROPOSITION 5.2. *For certain constants $c_j(z)$ depending only on z one has* $\dfrac{d^m g_z(x)}{dx^m} = g_z(x) \sum_{j=0}^{m} c_j(z) \tan h^j x$, $m = 0, 1, 2, 3, \ldots$ *Hence* $\left| \dfrac{d^m g_z(x)}{dx^m} \right| \leqslant M_m(z) e^{-(\mathrm{Re}\, z)|x|}$ *for constants* $M_m(z)$ *depending only on* m, z. *In particular if* $\mathrm{Re}\, z > 1/2$ *then* g_z *is a test function in the sense of Definition 3.2.*

The space $C_c^\infty(\mathbf{R})$ is contained in $S(\mathbf{R})$ and, similarly, can be assigned a topology — the so-called inductive limit topology — such that the inclusion map $C_c^\infty(\mathbf{R}) \to S(\mathbf{R})$ is continuous. A continuous linear functional T on $C_c^\infty(\mathbf{R})$ is called a *distribution.* A distribution T is a *tempered distribution* if T extends to a continuous linear functional (necessarily unique as $C_c^\infty(\mathbf{R})$ is dense in $S(\mathbf{R})$) on $S(\mathbf{R})$. Note that in terms of the seminorms $p_{n,m}$ on $S(\mathbf{R})$ defined by (5.1) a linear functional $\bar{T}$ on $S(\mathbf{R})$ is continuous $\leftrightarrow \exists c > 0$ and a finite non-empty subset F of the integers n, m such that $|\bar{T}(f)| \leqslant c \max_{n,m \in F} p_{n,m}(f) \,\forall f \in S(\mathbf{R})$.

We have noted earlier that a continuously differentiable compactly supported function on $\mathbf{R}$ is a test function. In particular each $g \in C_c^\infty(\mathbf{R})$ is a test function, and thus g plugs into Weil's formula (4.4). Define therefore a linear functional $T^W : C_c^\infty(\mathbf{R}) \to \mathbf{C}$ (Weil's distribution) via the left hand side of (4.4):

$$(5.3) \qquad T^W(g) = \sum_p g^*(p) \stackrel{\text{def.}}{=} \lim_{T \to \infty} \sum_{\substack{p \\ |\mathrm{Im}\, p| < T}} n_p\, g^*(p)$$

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[1]) That is, $(S(R), \tau)$ is a topological vector space and τ has a basis consisting of convex sets [4].

for $g \in C_c^\infty(G)$. J. Benedetto has shown in [2] that if the RH holds then T^W is tempered. Benedetto and D. Joyner have established the converse. Thus one has the following quite beautiful result.[1])

THEOREM 5.4 (Benedetto, Joyner). *The Riemann hypothesis holds* $\leftrightarrow$ *the Weil distribution* T^W *is tempered.*

Theorem 5.4 is preceded by the following more classical result.

THEOREM 5.5 (Weil [16]). *The Riemann hypothesis holds* $\leftrightarrow$ *the Weil distribution* T^W *is positive definite:* $T^W(g * g^0) \geqslant 0$ *for* $g \in C_c^\infty(\mathbf{R})$ *where* $g^0(x) \overset{\text{def.}}{=} \overline{g(-x)}$ *for* $x \in \mathbf{R}$.

Here $f_1 * f_2$ denotes the convolution of functions f_1, f_2:

$$(f_1 * f_2)(x) = \int_{\mathbf{R}} f_1(x-y) f_2(y) dy \,. \tag{5.6}$$

VI. THE MAIN TEST FUNCTION

Fix $z \in \mathbf{C}$ with $\operatorname{Re} z > 0$ and define g_z on $\mathbf{R}$ by

$$g_z(x) = (\cosh x)^{-z} \quad \text{for} \quad x \in \mathbf{R} \,. \tag{6.1}$$

By Proposition 5.2, $g_z \in S(\mathbf{R})$ and in fact g_z is a test function if $\operatorname{Re} z > \frac{1}{2}$ — the main test function which we shall consider. The author's motivation for considering g_z is as follows. If one is given a so-called connected rank 1 semisimple Lie group G (for example, $G = SL(2, \mathbf{R})$ in (1.1)) then using a so-called *Cartan decomposition* of G one can assign a *radial component* $t(x) \geqslant 0$ to each $x \in G$ and thus construct a function g_z on G by setting $g_z(x) = (\cosh t(x))^{-z}$. Hüber's formula (1.6) is obtained by plugging this function into Selberg's trace formula [6]; a side computation of the "spherical" Fourier transform of this g_z is needed.

We shall need, similarly, the Fourier transform of g_z in (6.1). Since $\cosh x \geqslant \frac{e^{|x|}}{2}$, $|g_z(x)| \leqslant 2^{\operatorname{Re} x} e^{-\operatorname{Re} z |x|}$ (which is Proposition 5.2 for $m = 0$ there). By (3.1) therefore (with $b = \operatorname{Re} z) \hat{g}_x$ is defined and holomorphic on $-\operatorname{Re} z < \operatorname{Im} s < \operatorname{Re} z$. Since g_z is an *even* function one has for $s = x \in \mathbf{R}$

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[1]) Added in proof: see [19].

(6.2) $$\hat{g}_z(x) = 2\int_0^\infty (\cos tx)(\cosh t)^{-z} dt$$

where by a table of integrals[1]), the latter integral is

$$\frac{2^{z-1}}{\Gamma(z)}\Gamma\left(\frac{z}{2}+\frac{xi}{2}\right)\Gamma\left(\frac{z}{2}-\frac{xi}{2}\right).$$

On the other hand the functions $s \to \Gamma\left(\frac{z}{2}+\frac{si}{2}\right), \Gamma\left(\frac{z}{2}-\frac{si}{2}\right)$ are holomorphic on $-\operatorname{Re} z < \operatorname{Im} s < \operatorname{Re} z$. We therefore get

PROPOSITION 6.3. *The complex Fourier transform $\hat{g}_z$ of g_z in (6.1) is given by*

(6.4) $$\hat{g}_z(s) = \frac{2^{z-1}}{\Gamma(z)}\Gamma\left(\frac{z}{2}+\frac{si}{2}\right)\Gamma\left(\frac{z}{2}-\frac{si}{2}\right)$$

for $-\operatorname{Re} z < \operatorname{Im} s < \operatorname{Re} z$ *(the domain on which $\hat{g}_z$ is defined and holomorphic); here* $\operatorname{Re} z > 0$; *see definition (3.1).*

VII. AN ANALOGUE OF HUBER'S FORMULA

In place now are all of the ingredients needed for the derivation of the main formula (Theorem 7.1) of the lecture — an analogue of formula (1.6). We derive it by plugging the function g_z into Weil's formula (say $z = s - 1/2$ with $\operatorname{Re} s > 1$ to guarantee, as pointed out, that g_z is a test function). Since g_z is even $\hat{g}_z$ is also even. By Proposition 6.3 $\hat{g}_z\left(\frac{i}{2}\right) + \hat{g}_z\left(\frac{-i}{2}\right) = 2\hat{g}_z\left(\frac{i}{2}\right)$ $= 2\frac{2^{z-1}}{\Gamma(z)}\Gamma\left(\frac{z}{2}-\frac{1}{4}\right)\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)$. Similarly, $g_z^*(p) = \hat{g}_z\left(i\left(\frac{1}{2}-p\right)\right)$ (definition (3.4)) $= \frac{2^{z-1}}{\Gamma(z)}\Gamma\left(\frac{z}{2}-\frac{1}{2}\left(\frac{1}{2}-p\right)\right)\Gamma\left(\frac{z}{2}+\frac{1}{2}\left(\frac{1}{2}-p\right)\right)$. For our $\hat{g}_z$ which is a nice Schwartz function the principal value $\lim_{T\to\infty}\int_{-T}^{T}$ in (4.4) can be replaced by the Lebesgue integral $\int_R$. Finally $g_z(\log n) \overset{\text{def.}}{=} (\cosh \log n)^{-z}$

[1]) [8], for example, page 506.

and $\dfrac{\Gamma\left(\dfrac{s-1}{2}\right)}{\Gamma\left(\dfrac{s+1}{2}\right)} = \dfrac{2}{s-1}$ so that Theorem 4.3 gives (for $z = s - 1/2$ with $\operatorname{Re} s > 1$ - i.e. $\operatorname{Re} z > 1/2$).

THEOREM 7.1. *Fix* $s \in \mathbf{C}$ *with* $\operatorname{Re} s > 1$. *Then in the notation of (4.1), (4.2)*

$$\frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \lim_{T\to\infty} \sum_{\substack{p \\ |\operatorname{Im} p| < T}} n_p \, \Gamma\left(\frac{s-p}{2}\right)\Gamma\left(\frac{s-(1-p)}{2}\right)$$

$$= \frac{\pi \cdot 2^{-s+\frac{3}{2}}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} (-2)\,[\cosh \log n]^{-s+\frac{1}{2}}$$

(7.2)
$$- \frac{\pi\Gamma\left(s-\frac{1}{2}\right)2^{-s+\frac{3}{2}}}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \log \pi + \frac{4\pi}{s-1}$$

$$+ \frac{\pi \cdot 2^{-s+\frac{3}{2}}}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \Gamma\left(s-\frac{1}{2}\right) \frac{1}{2\pi} \int_R \hat{g}_{s-\frac{1}{2}}(t) \operatorname{Re} \psi\left(\frac{1}{4}+\frac{it}{2}\right) dt$$

where Λ *is von Mangoldt's function (definition (2.4)) and*

$$\hat{g}_{s-\frac{1}{2}}(t) = \frac{2^{s-\frac{3}{2}}}{\Gamma\left(s-\frac{1}{2}\right)} \Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{it}{2}\right)$$

for $t \in \mathbf{R}$.

We note in regard to the integral in (7.2) one has

(7.3)
$$\frac{1}{2\pi}\int_R \hat{g}_{s-\frac{1}{2}}(t) \operatorname{Re} \psi\left(\frac{1}{4}+\frac{it}{2}\right) dt$$
$$= \frac{1}{2\pi}\int_R \hat{g}_{s-\frac{1}{2}}(t)\, \psi\left(\frac{1}{4}+\frac{it}{2}\right) dt$$

since $\hat{g}_{s-\frac{1}{2}}$ is even and since $\overline{\psi(s)} = \psi(\bar{s})$. In fact if ϕ is an even function

$$\int_{-T}^{T} \phi(t) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{it}{2}\right) dt = \frac{1}{2}\int_{-T}^{T} \phi(t)\left[\psi\left(\frac{1}{4} + \frac{it}{2}\right) + \psi\left(\frac{1}{4} - \frac{it}{2}\right)\right] dt$$

where by the change of variables

$$t \to -t, \int_{-T}^{T} \phi(t)\psi\left(\frac{1}{4} - \frac{it}{2}\right) dt = \int_{-T}^{T} \phi(-t)\psi\left(\frac{1}{4} + \frac{it}{2}\right) dt = \int_{-T}^{T} \phi(t)\psi\left(\frac{1}{4} + \frac{it}{2}\right) dt .$$

On the other hand by page 148 of Barner's paper [1]

THEOREM 7.4. *For* $a, b > 0, g \in S(\mathbf{R})$

(7.5) $$\frac{1}{2\pi}\int_R \hat{\check{g}}(t)\psi\left(a + \frac{it}{b}\right) dt = \int_0^\infty \left[\frac{g(0)}{x} - \frac{be^{(1-a)bx}}{1-e^{-bx}} g(-x)\right] e^{-bx}dx \text{ where}$$

$\check{g}(t) = g(-t)$ for $t \in \mathbf{R}$.

Given equation (7.3) we therefore have

PROPOSITION 7.6. *In formula (7.2)*

$$\frac{1}{2\pi}\int_R \hat{g}_{s-\frac{1}{2}}(t) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{it}{2}\right) dt$$

(7.7) $$= \frac{1}{2\pi}\int_R \hat{g}_{s-\frac{1}{2}}(t)\psi\left(\frac{1}{4} + \frac{it}{2}\right) dt$$

$$= \int_0^\infty \left[\frac{1}{x} - \frac{2e^{3/2x}}{1-e^{-2x}}(\cosh x)^{\frac{1}{2}-s}\right] e^{-2x}dx .$$

Note that in the sum $\sum_p$ in formula (7.2), p "corresponds" to the zero s_j^+ of Selberg's zeta function in formula (1.6) and $1 - p$ (which is also a zero of ζ by the functional equation in (ii) of Theorem 2.2) corresponds to the zero s_j^-.

As an application take $s = 3/2$ in Theorem 7.1. By Legendre's duplication formula

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\Gamma(s) = 2^{-1/2}\sqrt{\pi}\frac{\sqrt{\pi}}{2} = \frac{\pi}{2\sqrt{2}}\,.$$

In the functional equation $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ choose $z = 3/4 - p/2 = \dfrac{s-p}{2}$ so that

$$1 - z = \frac{1}{4} + \frac{p}{2} = \frac{s-(1-p)}{2} \to \Gamma\left(\frac{s-p}{2}\right)\Gamma\left(\frac{s-(1-p)}{2}\right) = \frac{\pi}{\sin \pi z}$$

where

$$\sin\left(\frac{3\pi}{4} - \frac{\pi p}{2}\right) = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi p}{2} + \sin\frac{\pi p}{2}\right).$$

For $s = 3/2$ formula (7.2) in conjunction with Proposition 7.6 therefore reduces to

$$\pi \lim_{T\to\infty} \sum_{\substack{p \\ |\mathrm{Im}\, p| < T}} \frac{n_p}{\cos\dfrac{\pi p}{2} + \sin\dfrac{\pi p}{2}}$$

$$= -2\sqrt{2}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \frac{n}{n^2+1} - \frac{\log \pi}{\sqrt{2}} + 2\pi$$

$$+ \frac{1}{\sqrt{2}} \int_0^{\infty} \left[\frac{1}{x} - \frac{2e^{3x/2}}{1-e^{-2x}} (\cosh x)^{-1}\right] e^{-2x}\, dx\,.$$

Via the change of variables $x = t/4$ the latter integral I is

$$\int_0^{\infty} \left[\frac{4}{t} - \frac{2e^{3t/8}\,2}{(1-e^{-t/2})\,(e^{t/4}+e^{-t/4})}\right] e^{-t/2}\frac{dt}{4}$$

(7.9)
$$= \int_0^{\infty} \left[\frac{e^{-t/2}}{t} - \frac{e^{-t/8}}{e^{t/4} - e^{-\frac{3}{4}t}}\right] dt$$

$$= \int_0^{\infty} \left[\frac{e^{-t/2}}{t} - \frac{e^{-\frac{3}{8}t}}{1-e^{-t}}\right] dt\,.$$

Thus by page 332 of [8], $I = -\log\frac{1}{2} + \psi\left(\frac{3}{8}\right)$, and by (7.8) we get

THEOREM 7.10. *In the notation of (4.1), (4.2)*

$$\pi \lim_{T\to\infty} \sum_{\substack{p \\ |\mathrm{Im}\, p| < T}} \frac{np}{\cos\frac{\pi p}{2} + \sin\frac{\pi p}{2}}$$

$$= -2\sqrt{2}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\frac{n}{n^2+1} - \frac{\log\pi}{\sqrt{2}} + 2\pi + \frac{1}{\sqrt{2}}[\log 2 + \psi(3/8)] .$$

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