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REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY

by Jeffrey SHALLIT

ABSTRACT. Real numbers with bounded partial quotients in their continued fraction expansion appear in many different fields of mathematics and computer science: Diophantine approximation, fractal geometry, transcendental number theory, ergodic theory, numerical analysis, pseudo-random number generation, dynamical systems, and formal language theory. In this paper we survey some of these applications.

1. INTRODUCTION AND DEFINITIONS

If x is a real number, we can expand x as a *simple continued fraction*

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

which we abbreviate in this paper as

$$x = [a_0, a_1, a_2, a_3, \ldots] \ .$$

In this paper, we only discuss the case of *regular continued fractions,* where a_0 is an integer and a_i, is a positive integer for $i \geqslant 1$; the expansion may or may not terminate. (For an introduction to continued fractions, see Hardy and Wright [135, Chap. 10]; for a more definitive work, see Perron [236]. For a history of continued fractions, see Brezinski [44].)

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If x is rational, then its continued fraction expansion terminates, and we can write $x = [a_0, a_1, \ldots, a_n]$. If we agree that $a_n = 1$ and $n \geqslant 1$, then this expansion is unique and we define

$$K(x) = \max_{1 \leqslant k \leqslant n} a_k ,$$

the largest partial quotient in the continued fraction for x.

If x is irrational, then its continued fraction expansion does not terminate. This expansion is unique. We write $x = [a_0, a_1, a_2, \ldots]$ and define

$$K(x) = \sup_{k \geqslant 1} a_k .$$

If $K(x) < \infty$, then we say that x has *bounded partial quotients*.

We define $\mathscr{B}_k = \{x \in \mathbf{R} \mid K(x) \leqslant k\}$, and $\mathscr{B} = \{x \in \mathbf{R} \mid K(x) < \infty\}$. Furthermore, let $\mathscr{E}_k = \mathscr{B}_k \cap (0, 1)$ and $\mathscr{E} = \mathscr{B} \cap (0, 1)$.

Real numbers with bounded partial quotients appear in many fields of mathematics and computer science: Diophantine approximation, fractal geometry, transcendental number theory, ergodic theory, numerical analysis, pseudo-random number generation, dynamical systems, and formal language theory. In this paper we survey some of these applications. Because of limited space, we cannot include a discussion of every result in detail. However, we have tried to include as complete a list of references as possible for those topics directly related to the main subject. Readers who know of other references are urged to contact the author (and provide a copy of the relevant paper, if possible). It is hoped that the list of references may contain some surprises even for experts in the field.

The author's interest in the subject arose from the material in Section 9. Because of this, the viewpoint presented in this article may be somewhat idiosyncratic.

2. Numbers of Constant Type

Let θ be an irrational number, and let $\| \theta \|$ denote the distance between θ and the closest integer.

Let $r \geqslant 1$ be a real number. We say that θ is *of type* $< r$ if

$$q \| q\theta \| \geqslant \frac{1}{r}$$

for all integers $q \geqslant 0$. Then we have the following

THEOREM 1. *If* θ *is of type* $< r$, *then* $K(\theta) < r$. *If* $K(\theta) = r$, *then* θ *is of type* $< r + 2$.

For a proof, see Baker [20, p. 47] or Schmidt [272, p. 22].

If there exists an $r < \infty$ such that θ is of type $< r$, then θ is said to be of *constant type*. By the theorem, numbers of constant type and numbers with bounded partial quotients coincide, and we will use these terms interchangeably in what follows.

A classical theorem of Lagrange states that the continued fraction for x is ultimately periodic if and only if x is a real quadratic irrational, and so all real quadratic irrationals are of constant type; see, for example, Lagrange [178] or Hardy and Wright [135, Chapter 10]. We will not explicitly discuss quadratic irrationals further in this paper.

Since

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$$

(see Cotes [58] and Euler [102]), we see that e is *not* of constant type. It is also known that the numbers $e^{2/n}$ and $\tan 1/n$ (n an integer $\geqslant 1$) are not of constant type. The status of π and γ (Euler's constant) is presently unknown. In section 9 we will see some explicit examples of transcendental numbers of constant type.

One way to interpret Theorem 1 is to say that numbers with bounded partial quotients are *badly approximable* by rationals; this term is also used frequently in the literature.

Note that if θ is of constant type, and r is a rational number, then $r\theta$ is also of constant type [54]. In fact, it is not hard to prove the following: let $r = a/b$ be a rational number, and suppose $K(\theta) = n$. Then $K(r\theta) \leqslant |ab|(n+2)$, and $K(\theta + r) \leqslant b^2(n+2)$; see Cusick and Mendès France [69].

From this, it follows that if a, b, c, d are integers with $ad - bc \neq 0$, then

$$\frac{a\theta + b}{c\theta + d}$$

has bounded partial quotients iff θ does. (See Shallit [278]. I would like to thank J. C. Lagarias for bringing this to my attention.) One can also deduce this result directly from the continued fraction, using results of Raney [256]. For another view of Raney's results, see van der Poorten [246].

Another related concept is the *Lagrange-Markoff* constant, denoted by $\mu(\theta)$. It is defined as follows:

$$\mu(\theta)^{-1} = \liminf_{q \to \infty} q \, \| q\theta \| .$$

Hurwitz [150] showed, among other things, that $\mu(\theta) \geqslant \sqrt{5}$; furthermore, $\mu\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$. Perron [234] showed that if

$$\theta = [a_0, a_1, a_2, \ldots] ,$$

then

$$\mu(\theta) = \limsup_{i \to \infty} ([a_{i+1}, a_{i+2}, a_{i+3}, \ldots] + [0, a_i, a_{i-1}, \ldots, a_1]) .$$

From this it follows that $\mu(\theta) < \infty$ if and only if θ is of constant type.

The range of $\mu(\theta)$, as θ takes on all irrational values, is known as the *Lagrange spectrum* and has been extensively studied. We direct the reader to the work of Lagrange [178, pp. 26-27]; Markoff [203, 204]; Heawood [138]; Perron [235]; Vinogradov, Delone, and Fuks [295]; Freiman [111]; Kinney and Pitcher [166]; Berštein [29]; Davis and Kinney [78]; Cusick [59, 62]; Flahive [117]; Cusick and Mendès France [69]; Wilson [301]; Dietz [89]; Pavone [232]; Prasad [249]; and especially the books of Koksma [172] and Cusick and Flahive [67].

For more on approximation by rational numbers, see Cassels [52], Schmidt [272], Kraaikamp and Liardet [313], Larcher [312].

3. The Metric Theory of Continued Fractions

Recall that $\mathscr{E}$ denotes the set of real numbers in (0, 1) with bounded partial quotients.

While it is easy to see $\mathscr{E}$ has uncountably many elements, nevertheless "most" numbers do *not* have bounded partial quotients. More precisely, we have the following

THEOREM 2 (Borel-Bernstein). *$\mathscr{E}$ is a set of measure* 0.

The theorem is due to Borel [38]. The original proof was not complete, as discussed in Bernstein [27]; further details were provided in a later paper of Borel [39]. For other proofs, see Hardy and Wright [135, Thm. 196] or Khintchine [160]. Also see Dyson [96].

Here is a sketch of a more general theorem: first, let us equate probability with Lebesgue measure, and assume x is a real number in (0, 1). Then, expanding x as a continued fraction, we have

$$x = [0, a_1, a_2, a_3, \ldots] ,$$

and we can consider each $a_i = a_i(x)$ to be a function of x. Then it is not difficult to show that

$$\Pr[a_n(x) = k] = \Theta\left(\frac{1}{k^2}\right).$$

From this, it follows that

$$\Pr[a_n(x) \geqslant k] = \Theta\left(\frac{1}{k}\right).$$

If the random variables $a_i(x)$ were *independent,* it would follow from the Borel-Cantelli lemma that

$$\Pr[a_n(x) \geqslant b_n \text{ infinitely often}] = 1$$

if and only if $\sum_{n \geqslant 1} \frac{1}{b_n}$ diverges. Unfortunately, the $a_i(x)$ are not independent, but they are "almost" independent; with some additional work, the result can be shown.

Now taking, e.g., $b_n = n$, we see that for almost all x, we have $a_n(x) \geqslant n$ infinitely often, and hence $\mathscr{E}$ is a set of measure 0.

Theorem 2 is a simple result in the *metric theory of continued fractions,* which had its origins in an 1812 letter from Gauss to Laplace. Gauss essentially stated [116] that

$$\lim_{n \to \infty} \Pr[a_n(x) = k] = \log_2\left(1 + \frac{1}{k(k+2)}\right),$$

and this was proven by Kuzmin [174, 175] and Lévy [186], independently.

Actually, even more is known. For example, Khintchine [160, 162] proved that if $f(n)$ is a non-negative function that does not grow too quickly, then with probability 1 we have

$$\lim_{m \to \infty} \frac{1}{m} \sum_{1 \leqslant k \leqslant m} f(a_k) = \sum_{r \geqslant 1} f(r) \log_2\left(1 + \frac{1}{r(r+2)}\right).$$

Now setting $f(i) = 1$ if $i = n$, and $f(i) = 0$ otherwise, we see that with probability 1, the fraction $\log_2\left(1 + \frac{1}{r(r+2)}\right)$ of the partial quotients in the continued fraction expansion of a real number x are equal to r.

Some early papers discussing the distribution of partial quotients include Gyldén [123, 124]; Brodén [45]; and Wiman [302].

For the classical metric theory of continued fractions, see (in addition to the papers mentioned above) Lévy [187, 188, 189, 191]; Khintchine [162, 163]; and Denjoy [83, 84, 85]. For more recent improvements, see Szüsz [289, 290]; de Vroedt [296]; Wirsing [303]; Rieger [261]; Babenko [12]; and Babenko and Jur'ev [13].

A more modern approach derives these results using powerful methods of ergodic theory. A good introduction is the book of Billingsley [31]. Other articles include Knopp [168]; Doeblin [91]; Ryll-Nardzewski [266]; Hartman, Marczewski, and Ryll-Nardzewski [137]; Hartman [136]; Lévy [190]; Rényi [257]; de Vroedt [297]; Stackelberg [283]; Šalát [267]; Philipp [239, 240, 241, 242, 243]; Philipp and Stackelberg [244]; and Galambos [112, 113, 114].

4. Continued Fractions for Algebraic Numbers

A major open problem is to determine if any algebraic numbers of degree > 2 are in $\mathscr{B}$. As Khintchine [164, 165, 160] has remarked,

> It is interesting to note that we do not, at the present time, know the continued-fraction expansion of a single algebraic number of degree higher than 2. We do not know, for example, whether the sets of elements [partial quotients] in such expansions are bounded or unbounded. In general, questions connected with the continued-fraction expansion of algebraic numbers of higher degree than the second are extremely difficult and have hardly been studied.

(The problem goes back at least to 1949, with the appearance of Khintchine's book [164]. The paragraph above most likely also appeared in the first (1936) edition of Khintchine's book, but I have not been able to verify this by examining a copy. I do not know any earlier explicit reference to the problem. A remark similar to Khintchine's was made by Delone in a foreword to a translation of Delone and Fadeev [82, p. iv].)

Khintchine's remark is still true today; there are only a few papers that have explicitly discussed the partial quotients of algebraic numbers of degree > 2. See, for example, Davenport[1]) [76]; Orevkov [229]; Pass [231]; Wolfskill [304]; Blinov and Rabinovich [34]; Bombieri and van der Poorten [37]; Dzenskevich and Shapiro [98]; and van der Poorten [247].

[1]) Actually, Davenport's results apply to all irrational numbers, not just algebraic numbers. Also see Mendès France [206].

One can deduce weak upper bounds on the growth of the partial quotients of algebraic numbers from results in Diophantine approximation. Suppose there exist constants C, s such that

$$\| q\theta \| > \frac{C}{q^s}$$

for all positive integers q. Wolfskill [304] remarked that the partial quotients a_i in the continued fraction expansion of θ then satisfy $a_i < A^{(s+\varepsilon)^i}$, where A depends on C and ε. Thus upper bounds can be deduced from the results in the following papers: Liouville [192]; Thue [291]; Siegel [280, 281]; Dyson [97]; Roth [265]; Davenport and Roth [77]; Baker [18, 19]; Feldman [105]; Bombieri [35]; Bombieri and Mueller [36]; Chudnovsky [55]; Easton [99]; and Baker and Stewart [22]. Stronger results were given by Davenport and Roth [77]. They showed that the denominators q_i of convergents to a real algebraic number θ satisfy

$$\log \log q_n < \frac{Cn}{\sqrt{\log n}} ;$$

here C is a constant that depends on θ but not on n. Furthermore, this constant can be made effective.

There are several methods known for computing the partial quotients for a given algebraic number. See the papers of Lagrange [177]; Vincent [294]; Cantor, Galyean, and Zimmer [50]; Churchhouse [56]; Rosen and Shallit [264]; Akritas and Ng [6, 7]; Thull [292]; and Akritas [1, 2, 3, 4, 5].

In 1769, Lagrange [177] showed that the real zero of $x^3 - 2x - 5$ has a continued fraction expansion which begins

$$[2, 10, 1, 1, 2, 1, 3, 1, 1, 12, \ldots] .$$

For some other explicit computations of the continued fraction expansions of algebraic numbers of degree > 2, see von Neumann and Tuckerman [217]; Richtmyer, Devaney, and Metropolis [260]; Bryuno [46]; Lang and Trotter [181]; Richtmyer [259]; and Pethö [238]. In 1964, J. Brillhart found that the real zero of $x^3 - 8x - 10$ had some unusually large partial quotients. An explanation was provided later by Churchhouse and Muir [57] and Stark [284].

5. Certain Sums in Diophantine Approximation

Let us agree to write $\{\theta\}$ for the fractional part of θ, namely, $\theta - [\theta]$.

One of the earliest appearances of real numbers with bounded partial quotients is in the theory of Diophantine approximation.

For example, consider the sum

$$s'_n(\theta) = \sum_{1 \leqslant k \leqslant n} \left(\{k\theta\} - \frac{1}{2}\right).$$

Clearly $s'_n(\theta) = O(n)$; but Lerch proved in 1904 that if θ has bounded partial quotients, we have $s'_n(\theta) = O(\log n)$. See [184]. (This result was also announced by Hardy and Littlewood in 1912; see [128].)

At the International Congress of Mathematicians in 1912, Hardy and Littlewood [128] announced several theorems on Diophantine approximation, some of which relate to the subject at hand. For example, they defined

$$s_n(\theta) = \sum_{1 \leqslant k \leqslant n} e^{\left(k - \frac{1}{2}\right)^2 \pi i \theta},$$

and stated that if θ has bounded partial quotients, then $s_n(\theta) = O(\sqrt{n})$. The proof appeared later; see [130].

At the same Congress, Hardy and Littlewood announced that

$$\sum_{1 \leqslant k \leqslant n} \left(\{k\theta\} - \frac{1}{2}\right)^2 = \frac{n}{12} + O(1),$$

for all irrational θ. This is incorrect, and the correct formulation was stated in a 1922 paper: the result holds for many, but not all irrationals, and in particular it holds for θ with bounded partial quotients. See [132] for the statement and [133] for a proof.

Hardy and Littlewood also examined other series of interest. They defined:

$$U_n(\theta) = \sum_{1 \leqslant k \leqslant n} \frac{(-1)^k}{k \sin k\pi\theta},$$

$$V_n(\theta) = \sum_{1 \leqslant k \leqslant n} \frac{(-1)^k}{\sin k\pi\theta},$$

and

$$W_n(\theta) = \sum_{1 \leqslant k \leqslant n} \frac{1}{(\sin k\pi\theta)^2}.$$

They showed that if θ has bounded partial quotients, then $U_n(\theta) = O(\log n)$, $V_n(\theta) = O(n)$, and $W_n(\theta) = O(n^2)$. See [134].

(Warning to the reader: in their papers, Hardy and Littlewood used the notation $\{x\}$ to mean $x - [x] - \frac{1}{2}$, not $x - [x]$, as is more standard today.)

For other related papers, see Hardy and Littlewood [129, 131]; the collected works of Hardy [127]; Ostrowski [230]; Khintchine [161]; Oppenheim [228]; Chowla [53, 54]; Walfisz [298, 299, 300], and Schoissengeier [314].

Others researchers have examined similar sums in connection with numbers with bounded partial quotients. See the papers of Faĭziev [104] Ivanov [151], and Schoissengeier [274].

6. Fractal Geometry

Numbers with bounded partial quotients provided an early example of a set with non-integral Hausdorff dimension.

Let $\dim S$ denote the Hausdorff dimension of the set S (for a definition, see, e.g. Falconer [103]). We use the definitions of $\mathscr{E}$ and $\mathscr{E}_k$ from section 1.

In 1928, Jarník [152] proved that $\dim \mathscr{E} = 1$,

$$\frac{1}{4} < \dim \mathscr{E}_2 < 1 \, ,$$

and

$$1 - \frac{4}{k \log 2} < \dim \mathscr{E}_k < 1 - \frac{1}{8k \log k} \, ,$$

for $k > 8$. An exposition of Jarník's work can be found in Rogers [263].

In 1941, Good proved the following result [118]:

$$\dim \mathscr{E}_k = \lim_{n \to \infty} \sigma_{k,n} \, ,$$

where $\sigma = \sigma_{k,n}$ is the real root of the equation

$$\sum_{1 \leqslant a_1, a_2, \ldots, a_n \leqslant k} Q(a_1, a_2, \ldots, a_n)^{-2\sigma} = 1$$

and $Q(\,)$ denotes Euler's continuant polynomial. (These are multivariate polynomials, defined by $Q(\,) = 1$, $Q(a_1) = a_1$, and

$$Q(a_1, a_2, \ldots a_n) = a_n Q(a_1, a_2, \ldots, a_{n-1}) + Q(a_1, a_2, \ldots, a_{n-2})$$

for $n \geqslant 2$.)

Good also obtained the estimate $.5306 < \dim \mathscr{E}_2 < .5320$. This was improved by Bumby [48] in 1985 to $.5312 \leqslant \dim \mathscr{E}_2 \leqslant .5314$. More recently, Hensley [140] showed that $.53128049 < \dim \mathscr{E}_2 < .53128051$. For other results on the Hausdorff dimension of $\mathscr{E}_k$ and related sets, see Jarník [153]; Besicovitch [30]; Rogers [262]; Baker and Schmidt [21]; Hirst [147, 148]; Billingsley and Henningsen [32]; Cusick [63, 64, 65]; Pollington [245]; Kaufman [158]; Marion [202]; Gardner and Mauldin [115]; Ramharter [253, 254]; and Hensley [139, 141, 308, 309].

7. Schmidt's Game

W. M. Schmidt [270] introduced the following two-player game, called an (α, β) game: let α, β be real numbers with $0 < \alpha, \beta < 1$. First Bob chooses a closed interval on the real line, called B_1. Then Alice chooses a closed interval $A_1 \subset B_1$, such that the length of A_1 is α times the length of B_1. Then Bob chooses a closed interval $B_2 \subset A_1$, such that the length of B_2 is β times the length of A_1, and so on. If the intersection of all the intervals A_i is a number with bounded partial quotients, then Alice is declared the winner; otherwise Bob is declared the winner.

Schmidt showed that if $0 < \alpha < 1/2$, then Alice always has a winning strategy for this game. This is somewhat surprising, since as we have seen above, the set $\mathscr{E}$ of numbers with bounded partial quotients has Lebesgue measure 0.

Using the theory of (α, β) games, Schmidt also reproved the result of Jarník that $\mathscr{E}$ has Hausdorff dimension 1.

Several papers have proved other results on (α, β) games: see Schmidt [271]; Freiling [109, 110]; and Dani [70, 71, 72]. Also see Schmidt [272, Chapter 3].

8. Hall's theorem

If S and T are sets, then by $S + T$ we mean the set

$$\{s + t \mid s \in S, t \in T\} .$$

Similarly, by $S \cdot T$ we mean the set

$$\{st \mid s \in S, t \in T\} .$$

If S is a set of Lebesgue measure zero, then it is quite possible for $S + S$ to have positive measure. For example, if C denotes the Cantor set (numbers in [0, 1] containing only 0's and 2's in their ternary expansion), then C has measure 0, and it is not hard to show that $C + C = [0, 2]$; see Borel [40] or Pavone [233]. The result is due to Steinhaus [310]; I am most grateful to G. Myerson for bringing this to my attention.

As we have seen above, the set $\mathscr{B}$, and hence each $\mathscr{B}_k$, also has Lebesgue measure zero. In 1947 Hall proved the following theorem [126]:

THEOREM 3. *Every real number* x *can be written as* $x = y + z$, *where* $y, z \in \mathscr{B}_4$. *Every real number* $x \geqslant 1$ *can be written as* $x = yz$, *where* $y, z \in \mathscr{B}_4$.

An exposition of Hall's result can be found in Cusick and Flahive [67].

Using the notation of the first paragraph of this section, we could rephrase the statement of Hall's theorem as follows: $\mathscr{B}_4 + \mathscr{B}_4 = \mathbf{R}$, and $[1, \infty) \subseteq \mathscr{B}_4 \cdot \mathscr{B}_4$.

In 1973, Cusick [61] proved that $\mathscr{B}_3 + \mathscr{B}_3 + \mathscr{B}_3 = \mathbf{R}$, and $\mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 = \mathbf{R}$. He also observed that $\mathscr{B}_3 + \mathscr{B}_3 \neq \mathbf{R}$, and $\mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 \neq \mathbf{R}$. These results were independently discovered by Diviš [90] and J. Hlavka [1]) [149]. Hlavka also showed that $\mathscr{B}_3 + \mathscr{B}_4 = \mathbf{R}$, and similar results. Apparently the status of $\mathscr{B}_2 + \mathscr{B}_5$ and $\mathscr{B}_2 + \mathscr{B}_6$ is still open.

For results of a similar character, see Cusick [60]; Cusick and Lee [68]; and Bumby [47].

9. EXPLICIT EXAMPLES OF TRANSCENDENTAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS

In Lang [179] we find the following statement:

> No simple example of [irrational] numbers of constant type, other than the one given above [real quadratic irrationals], is known. The best guess is that there are no other "natural" examples.

(Also see Lang [180].)

However, in 1979 Kmošek [167] and Shallit [275] independently discovered the following "natural" example of numbers of constant type.

THEOREM 4. *Let* $n \geqslant 2$ *be an integer and define*

(1) $$f(n) = \sum_{i \geqslant 0} n^{-2^i} .$$

---

[1]) Note this is *not* same person as E. Hlawka!

Then $K(f(2)) = 6$ *and* $K(f(n)) = n + 2$ *for* $n \geqslant 3$.

For example, we have

$$f(3) = [0, 2, 5, 3, 3, 1, 3, 5, 3, 1, 5, 3, 1, \ldots] .$$

It is also possible to show that $K(nf(n)) = n$.

For related articles, see Köhler [171]; Pethö [237]; Shallit [277], and Wu [305]. (An aside: Mignotte [213] proved that there exists a constant c such that

$$\left| f(2) - \frac{p}{q} \right| > \frac{c}{q^3}$$

for all integers p and odd q. However, by combining Theorems 1 and 4, we get the improved bound

$$\left| f(2) - \frac{p}{q} \right| > \frac{1}{8q^2}$$

for all integers $q \geqslant 1$. Also see Derevyanko [86].)

Kempner [159] had proved in 1916 that $f(n)$ is transcendental for all integers $n \geqslant 2$. Mahler [200] also proved this result; also see Loxton and van der Poorten [195].

(Kempner seems to be responsible for a mistake that has been perpetuated in several papers. He called the series in Eq. (1) above the *Fredholm series,* in the belief that Fredholm studied it. Kempner referred to a paper of Mittag-Leffler [215], but this paper discusses the series

$$\sum_{i \geqslant 0} x^{i^2} ,$$

which is very different. An examination of Fredholm's collected works [108] did not turn up any papers on the series in Eq. (1). This mistaken attribution was repeated by Schneider in his classic work on transcendental numbers [273], and then repeated by other authors; see, e.g. Pethö [237]; Mendès France [207].)

Mendès France pointed out an intriguing connection between the continued fraction expansion of $f(n)$ and iterated paperfolding, which we now describe briefly.

If we fold a piece of paper in half repeatedly, say n times, always folding right hand over left hand, we get a series of $2^n - 1$ hills and valleys upon unfolding. Let us denote the hills by $+1$ and the valleys by -1. Letting X_n be the sequence of folds so obtained, it is not hard to see that

$$X_{n+1} = X_n \quad (+1) \quad -X_n^R ,$$

where juxtaposition denotes concatenation, and by X_n^R we mean the sequence X_n taken in reverse order.

More generally, we can choose to introduce a hill or valley at the nth fold. If we denote the nth fold by a_n, then after folding with $a_1, a_2, \ldots, a_n$, upon unfolding we get the sequence

$$F_{a_1}(F_{a_2}(\cdots(F_{a_n}(\varepsilon))\cdots)),$$

where ε denotes a sequence of length 0, and F_i is the *folding map,* given by

$$F_i(X) = X \quad i \quad -X^R .$$

Mendès France observed that the continued fraction expansion of $f(n)$ could be written in terms of the folding map F_i; see Mendès France [207]; Blanchard and Mendès France [33]; Dekking, van der Poorten and Mendès France [80]; Shallit [276]; and Mendès France and Shallit [209].

More recently, van der Poorten and Shallit [248] discovered a closer connection between paperfolding and continued fractions. Suppose we consider the formal power series

$$g(X) = \sum_{k \geqslant 0} X^{-2^k} \in \mathbf{Q}((1/X)) .$$

Then $Xg(X)$ can be expanded as a continued fraction, and it is not hard to prove that

$$Xg(X) = [1, F_{-X}(F_{-X}(\cdots(F_{-X}(X))\cdots))] ;$$

i.e. the continued fraction is given by the iterated folding of a piece of paper!

Using this result, we can prove the following theorem: let $\varepsilon_0 = 1$ and $\varepsilon_i = \pm 1$ for $i \geqslant 1$. Then the continued fraction expansion of each of the numbers

$$2 \sum_{i \geqslant 0} \varepsilon_i 2^{-2^i}$$

consists solely of 1's and 2's. For example,

$$2f(2) = [1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, \ldots] .$$

Let us now turn to other constructions of transcendental numbers with bounded partial quotients.

Since the set $\mathscr{B}$ is uncountable, while the set of algebraic numbers is countable, it is clear that almost all elements of $\mathscr{B}$ are transcendental. However, many investigators were concerned with the *explicit* construction of transcendental elements of $\mathscr{B}$. For example, Baker proved that

$$[0,\ 1,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ \overbrace{2,\ \ldots,\ 2}^{8},\ \overbrace{1,\ \ldots,\ 1}^{16},\ \overbrace{2,\ \ldots,\ 2}^{32},\ \ldots]$$

and similar numbers are transcendental; see [16]. Previously, Maillet had given similar examples, but not explicitly [201]. Other examples have been recently given by Davison [79]. Also see Grant [120].

10. "Quasi-Monte-Carlo" Methods and Zaremba's Conjecture

In this section we briefly discuss some integration methods that depend on rational numbers with small partial quotients. There is a large literature on this subject; the interested reader can start with the comprehensive survey of Niederreiter [220].

(This section is tied to the main subject in the following manner: we wish to construct explicitly rational numbers with small partial quotients. One way to do this is to take an irrational number with bounded partial quotients and employ the sequence of convergents.)

In s-dimensional "quasi-Monte Carlo" integration, we approximate the integral

$$\int_{[0,\,1]^s} f(\mathbf{t})\,d\mathbf{t} \tag{2}$$

by the sum

$$\frac{1}{n} \sum_{1 \leqslant k \leqslant n} f(\mathbf{x}_k)\ ,$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is a set of points in $[0, 1]^s$.

The goal of quasi-Monte Carlo integration is to choose the points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ so as to minimize the error in the approximation.

In the method of *good lattice points,* we assume that the function f is periodic of period 1 in each variable. We choose a large fixed integer m and a special lattice point $\mathbf{g} \in \mathbf{Z}^s$. Then we approximate the integral (2) with the sum

$$\frac{1}{m} \sum_{1 \leqslant k \leqslant m} f\left(\frac{k}{m}\mathbf{g}\right).$$

"Good" lattice points $\mathbf{g}$ make the error in this approximation relatively small.

Let $\mathbf{h} = (h_1, h_2, \ldots, h_s)$ and define

$$r(\mathbf{h}) = \prod_{1 \leqslant i \leqslant s} \max(1, |h_i|) .$$

Also define

$$\rho(\mathbf{g}, m) = \min_{\mathbf{h}} r(\mathbf{h}) ,$$

where $\mathbf{h}$ ranges over all lattice points with

$$\frac{-m}{2} < h_j \leqslant \frac{m}{2} ,$$

$\mathbf{h} \neq 0$, and $\mathbf{h} \cdot \mathbf{g} \equiv 0 \pmod{m}$. It can be shown that good lattice points correspond to large values of ρ.

Now consider the 2-dimensional case, i.e. $s = 2$. Let $\mathbf{g} = (1, g)$ with $\gcd(g, m) = 1$. Then Zaremba [306] showed that

$$\frac{m}{K(g/m) + 2} \leqslant \rho(\mathbf{g}, m) \leqslant \frac{m}{K(g/m)} .$$

Hence good lattice points correspond to rationals g/m with small partial quotients.

For other connections with numerical integration, see the papers of Haber and Osgood [125] and Zaremba [307].

We now turn to Zaremba's conjecture. Define

$$Z(n) = \min_{\substack{1 \leqslant j \leqslant n \\ \gcd(j,n) = 1}} K\left(\frac{j}{n}\right) .$$

Then Zaremba [307] conjectured that $Z(n) \leqslant 5$.

Borosh [41] showed that Zaremba's conjecture is true for $1 \leqslant n \leqslant 10000$. In this range, only two integers have $Z(n) = 5$, namely $n = 54$ and $n = 150$. Twenty-five integers in this range have $Z(n) = 4$; the smallest is 6 and the largest is 6234. A brief discussion of Zaremba's conjecture up to 1978 can be found in [220].

Borosh and Niederreiter [42] suggested that in fact $Z(n) \leqslant 3$ for all sufficiently large n. The most extensive computation seems to be that of Knuth, cited in [42], which verified that $Z(n) \leqslant 3$ for $10000 \leqslant n \leqslant 3200000$.

Zaremba's conjecture is true for certain infinite sequences. For example, we certainly have $Z(F_k) = 1$ for $k \geqslant 1$, where F_k is the k-th Fibonacci number. It follows from the results of Kmošek and Shallit cited above that $Z(2^{2^k-1}) \leqslant 2$ for all $k \geqslant 0$.

Borosh and Niederreiter [42] showed that $Z(2^k) \leqslant 3$ for $6 \leqslant k \leqslant 35$.

More recently, Niederreiter [223] proved that Zaremba's conjecture holds for all powers of 2; in fact, we have $Z(2^k) \leqslant 3$ for all $k \geqslant 0$.

Larcher [182, Corollary 2] proved the existence of a constant c, such that for every $n \geqslant 1$ there exists a positive integer $j \leqslant n$, relatively prime to n, such that if

$$j/n = [0, a_1, a_2, \ldots, a_m] ,$$

then

$$\sum_{1 \leqslant i \leqslant m} a_i < c(\log n)\,(\log\log n)^2 .$$

This is close to the best possible bound $O(\log n)$, which was reportedly conjectured by L. Moser (although I do not know a reference); the bound would be a consequence of Zaremba's conjecture.

For other results connected with Zaremba's conjecture, see the papers of Cusick [63, 66]; Niederreiter [224]; Sander [268]; and Hensley [315].

11. Properties of the sequence $n\theta$ (mod 1)

If θ is a real number, by θ (mod 1) we mean $\{\theta\} = \theta - [\theta]$, the fractional part of θ.

It has been known at least since Bernoulli [26] that properties of the sequence $\theta, 2\theta, 3\theta, \ldots$ are intimately connected with the continued fraction expansion for θ. The distribution of $n\theta$ (mod 1) is a vast subject, and we restrict ourselves to mentioning several results connected with numbers of constant type.

Let θ be an irrational number, and let

$$0 = a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} = 1$$

be the sequence of points $\{k\theta\}$, $1 \leqslant k \leqslant n$, arranged in ascending order. Define

$$\delta_\theta(n) = \max_{1 \leqslant i \leqslant n+1} a_i - a_{i-1} .$$

Then Graham and van Lint [119] proved the following theorem:

$$\limsup_{n\to\infty} n\delta_\theta(n) < \infty$$

if and only if θ is a number of constant type.

Boyd and Steele [43] introduced the function $l_n^+(\theta)$, the length of the longest increasing subsequence of $\{\theta\}, \{2\theta\}, \ldots, \{n\theta\}$. They proved that

$$\liminf_{n\to\infty} \frac{l_n^+(\theta)}{\sqrt{n}} > 0$$

and

$$\limsup_{n\to\infty} \frac{l_n^+(\theta)}{\sqrt{n}} < \infty$$

if and only if the partial quotients of θ are bounded.

For some other results on $\{n\theta\}$ connected with bounded partial quotients, see Ennola [100, 101]; Lesca [185]; Drobot [92]; and Strauch [288].

12. Discrepancy and Dispersion

Let $\omega = (x_1, x_2, x_3, \ldots)$ be a sequence of real numbers. Let $I \subseteq [0, 1)$ be an interval and let $|I|$ denote its length. Define the counting function $S_n(I) = S_n(I, \omega)$ as the number of terms x_k, $1 \leqslant k \leqslant n$, for which $\{x_k\} \in I$.

The *discrepancy* $D_n(x_1, x_2, \ldots, x_n)$ is a measure of how much the sequence $x_1, x_2, \ldots, x_n$ deviates from a uniform distribution. It is defined as follows:

$$D_n(\omega) = D_n(x_1, x_2, \ldots, x_n) = \sup_{I \subseteq [0,1)} \left| \frac{S_n(I, \omega)}{n} - |I| \right| .$$

Now consider the discrepancy of the sequence $\omega = (\theta, 2\theta, 3\theta, \ldots)$. If θ has bounded partial quotients, then the discrepancy of ω is small. In particular, we have the following estimate: If $K(\theta) \leqslant k$, then

$$nD_n(\omega) \leqslant 3 + \left(\frac{1}{\log \alpha} + \frac{k}{\log(k+1)} \right) \log n$$

for $\alpha = \frac{1}{2}(1 + \sqrt{5})$. See, for example, Kuipers and Niederreiter [173].

For other results connecting discrepancy and the boundedness of the partial quotients, see the papers of Niederreiter [218] and Dupain and Sós [94, 95]. Also see Beck and Chen [25] and Richert [258].

We can also consider the so-called L^2 discrepancy, T_n, defined as follows: let

$$R_n(t) = \frac{S_n([0, t), \omega)}{n} - t$$

and put

$$T_n(\omega) = \left(\int_0^1 R_n^2(t)dt\right)^{1/2}.$$

It is possible to generalize the definitions of D_n and T_n to the multi-dimensional case, though we omit the details. By appealing to numbers with bounded partial quotients, Davenport [73] constructed sequences in two dimensions with low L^2 discrepancy. Also see Proinov [250, 251, 252].

Another measure connected with sequences is called *dispersion*. Let $\omega = (x_1, x_2, \ldots)$ and define the dispersion

$$d_n(\omega) = \sup_{x \in [0,1]} \min_{1 \leqslant k \leqslant n} |x - x_k|,$$

essentially half the distance between the most widely separated points of the sequence $x_1, x_2, \ldots, x_n$. (Compare with the function $\delta_\theta(n)$ in Section 11.)

Niederreiter [221] considered the dispersion of the sequence $\{n\theta\}$. He showed that if θ has bounded partial quotients, then $d_n(\omega) = O(1/n)$. He also gave a more detailed estimate, showing that $d_n(\omega)$ is approximately $K(\theta)/4n$. Also see Drobot [93] and Larcher [311].

13. Connections with Ergodic Theory

Let θ be irrational, $\omega = (\theta, 2\theta, \ldots)$ and $S_n(I, \omega)$ be defined as in the previous section. Veech [293] developed connections between S_n and ergodic theory. We mention one result that is number-theoretic in nature. Let $x_n = S_n(I, \omega) \bmod 2$, and define

$$\mu_\theta(I) = \lim_{n \to \infty} \frac{1}{n} \sum_{1 \leqslant k \leqslant n} x_k,$$

if the limit exists. Then Veech showed that $\mu_\theta(I)$ exists for all $I \subseteq [0, 1)$ if and only if the partial quotients of θ are bounded.

For other connections with ergodic theory, see the papers of Stewart [286]; del Junco [154]; Dani [70, 72]; and Baggett and Merrill [14, 15].

14. Pseudo-Random Number Generation

Lehmer [183] introduced the *linear congruential method* for pseudo-random number generation. Let X_0, m, a, c be given, and define

$$X_{k+1} = aX_k + c \pmod{m} ,$$

for $k \geqslant 0$. For this to be a good source of "random" numbers, we want the sequence X_k to be uniformly distributed, as well as the sequence of pairs (X_k, X_{k+1}), triples, etc.

A test for randomness called the *serial test* on pairs (X_k, X_{k+1}) amounts to the two-dimensional version of the discrepancy mentioned above in Section 12. This turns out to be essentially the function $\rho(\mathbf{g}, m)$ defined in Section 10. Thus linear congruential generators that pass the pairwise serial test arise from rationals a/m having small partial quotients in their continued fraction expansion. See the papers of Dieter [87, 88]; Niederreiter [219, 220, 222]; Knuth [170, Section 3.3.3]; and Borosh and Niederreiter [42].

15. Formal Language Theory

Let $w = w_0 w_1 w_2 \cdots$ be an infinite word over a finite alphabet. We say that the finite word $x = x_0 x_1 \cdots x_n$ is a *subword* of w if there exists $m \geqslant 0$ such that $w_{m+i} = x_i$, for $0 \leqslant i \leqslant n$. We say that w is *k-th power free* if x^k is never a subword of w, for all nonempty words x. Here is a classical example: let $s(n)$ denote the number of 1's in the binary expansion of n. Then the infinite *word of Thue-Morse*

$$t = t_0 t_1 t_2 \cdots = 0110100110010110 \cdots ,$$

defined by $t_n = s(n) \bmod 2$, is cube-free.

Another way to define infinite words is as the fixed point of a homomorphism on a finite alphabet. For example, the Thue-Morse word t is a fixed point of φ, where $\varphi(0) = 01$ and $\varphi(1) = 10$.

A famous infinite word which has been extensively studied is the *Fibonacci word*

$$f = 101101011011010 \cdots ;$$

it is a fixed point of the homomorphism μ, where $\mu(1) = 10$ and $\mu(0) = 1$. For some of the properties of this word, see the survey of Berstel [28]. Karhumäki showed that f is fourth-power-free; see [155].

Now we define some special infinite words. Let $\theta \in [0, 1)$ and define the infinite word $w = w_1 w_2 w_3 \cdots$ as follows:

$$w_n = [(n+1)\theta] - [n\theta] .$$

If we set $\theta = (\sqrt{5} - 1)/2$, we get the Fibonacci word f. Recently, Mignosi [212] proved the following theorem: there exists a k such that w is k-th power-free, if and only if θ has bounded partial quotients. (One direction of Mignosi's theorem follows easily from two different descriptions of w in terms of the continued fraction expansion for θ; see Markoff [205]; Stolarsky [287]; and Fraenkel, Mushkin, and Tassa [107].)

16. Other Results

Let θ be an irrational number of constant type. Let p_n/q_n denote the n-th convergent to θ.

For n a positive integer, let $P(n)$ denote the largest prime factor of n. Then given $\varepsilon > 0$, there exists a constant $c = c(\theta; \varepsilon)$ such that the number of positive integers $n \leqslant x$ with

$$P(q_n) < c \log \log q_n$$

is at most εx. This is a result of Shorey [279].

Schmidt [269] showed that if $f_1, f_2, \ldots$ is a sequence of differentiable functions whose derivatives are continuous and vanish nowhere, then there are uncountably many numbers θ such that all the numbers $f_1(\theta), f_2(\theta), \ldots$ have bounded partial quotients. For related results, see Davenport [74, 75] and Cassels [51].

Other topics connected with real numbers with bounded partial quotients not discussed in this survey include transcendental number theory (see Baker [17]; Flicker [106]; Bundschuh [49]; Angell [11]), Fibonacci hashing on digital computers (see Knuth [169, pp. 510-512]), dynamical systems and global analysis (see Deligne [81]; Katznelson [156]; Herman [142, 143, 144, 145, 146]; Meyer [211]; de la Llave [193, 194]; MacKay [196, 197]; MacKay, Meiss, and

Percival [198]; Greene and MacKay [121]; Gutierrez [122]; Rand [255]; Stark [285]; Katznelson and Ornstein [157]; MacKay and Stark [199]; Sinai and Khanin [282]), and in the proof of a theorem connected with Kemperman's inequality (see Laczkovich [176]). For a connection with the "entropy" of a curve, see Mendès France [208].

17. Related Results

In this survey, we have restricted our attention to *real numbers* with bounded partial quotients. However, we would be remiss to omit mentioning the work on formal power series over a finite field having partial quotients of bounded degree. See the papers of Baum and Sweet [23, 24]; Mills and Robbins [214]; Mesirov and Sweet [210]; Mullen and Niederreiter [216]; Niederreiter [225, 227, 226]; Allouche [8]; and Allouche, Mendès France, and van der Poorten [10].

It is perhaps appropriate to mention the following question of Mendès France, which asks (roughly) if a formal power series over a finite field is algebraic and the partial quotients in its continued fraction expansion are of bounded degree, then must those partial quotients be accepted by a finite automaton? For a more precise version of this conjecture, see the paper of Allouche, Betrema, and Shallit [9]. This paper also gives some examples for which the answer to Mendès France's question is positive. However, the partial quotients in the continued fraction for the power series of Baum and Sweet [23], which were later described explicitly by Mills and Robbins [214], do not seem to be accepted by a finite automaton.

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