Zeitschrift: L'Enseignement Mathématique

Band: 38 (1992)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY

Kapitel: 5. Certain Sums in Diophantine Approximation

Autor: Shallit, Jeffrey

DOI: https://doi.org/10.5169/seals-59489

5. Certain Sums in Diophantine Approximation

Let us agree to write $\{\theta\}$ for the fractional part of θ, namely, $\theta - [\theta]$.

One of the earliest appearances of real numbers with bounded partial quotients is in the theory of Diophantine approximation.

For example, consider the sum

$$s'_n(\theta) = \sum_{1 \leqslant k \leqslant n} \left(\{k\theta\} - \frac{1}{2}\right).$$

Clearly $s'_n(\theta) = O(n)$; but Lerch proved in 1904 that if θ has bounded partial quotients, we have $s'_n(\theta) = O(\log n)$. See [184]. (This result was also announced by Hardy and Littlewood in 1912; see [128].)

At the International Congress of Mathematicians in 1912, Hardy and Littlewood [128] announced several theorems on Diophantine approximation, some of which relate to the subject at hand. For example, they defined

$$s_n(\theta) = \sum_{1 \leqslant k \leqslant n} e^{\left(k - \frac{1}{2}\right)^2 \pi i \theta},$$

and stated that if θ has bounded partial quotients, then $s_n(\theta) = O(\sqrt{n})$. The proof appeared later; see [130].

At the same Congress, Hardy and Littlewood announced that

$$\sum_{1 \leqslant k \leqslant n} \left(\{k\theta\} - \frac{1}{2}\right)^2 = \frac{n}{12} + O(1),$$

for all irrational θ. This is incorrect, and the correct formulation was stated in a 1922 paper: the result holds for many, but not all irrationals, and in particular it holds for θ with bounded partial quotients. See [132] for the statement and [133] for a proof.

Hardy and Littlewood also examined other series of interest. They defined:

$$U_n(\theta) = \sum_{1 \leqslant k \leqslant n} \frac{(-1)^k}{k \sin k\pi\theta},$$

$$V_n(\theta) = \sum_{1 \leqslant k \leqslant n} \frac{(-1)^k}{\sin k\pi\theta},$$

and

$$W_n(\theta) = \sum_{1 \leqslant k \leqslant n} \frac{1}{(\sin k\pi\theta)^2}.$$

They showed that if θ has bounded partial quotients, then $U_n(\theta) = O(\log n)$, $V_n(\theta) = O(n)$, and $W_n(\theta) = O(n^2)$. See [134].

(Warning to the reader: in their papers, Hardy and Littlewood used the notation $\{x\}$ to mean $x - [x] - \frac{1}{2}$, not $x - [x]$, as is more standard today.)

For other related papers, see Hardy and Littlewood [129, 131]; the collected works of Hardy [127]; Ostrowski [230]; Khintchine [161]; Oppenheim [228]; Chowla [53, 54]; Walfisz [298, 299, 300], and Schoissengeier [314].

Others researchers have examined similar sums in connection with numbers with bounded partial quotients. See the papers of Faĭziev [104] Ivanov [151], and Schoissengeier [274].

6. Fractal Geometry

Numbers with bounded partial quotients provided an early example of a set with non-integral Hausdorff dimension.

Let $\dim S$ denote the Hausdorff dimension of the set S (for a definition, see, e.g. Falconer [103]). We use the definitions of $\mathscr{E}$ and $\mathscr{E}_k$ from section 1.

In 1928, Jarník [152] proved that $\dim \mathscr{E} = 1$,

$$\frac{1}{4} < \dim \mathscr{E}_2 < 1 ,$$

and

$$1 - \frac{4}{k \log 2} < \dim \mathscr{E}_k < 1 - \frac{1}{8k \log k},$$

for $k > 8$. An exposition of Jarník's work can be found in Rogers [263].

In 1941, Good proved the following result [118]:

$$\dim \mathscr{E}_k = \lim_{n \to \infty} \sigma_{k,n} ,$$

where $\sigma = \sigma_{k,n}$ is the real root of the equation

$$\sum_{1 \leqslant a_1, a_2, \ldots, a_n \leqslant k} Q(a_1, a_2, \ldots, a_n)^{-2\sigma} = 1$$

and $Q(\ )$ denotes Euler's continuant polynomial. (These are multivariate polynomials, defined by $Q(\ ) = 1$, $Q(a_1) = a_1$, and

$$Q(a_1, a_2, \ldots a_n) = a_n Q(a_1, a_2, \ldots, a_{n-1}) + Q(a_1, a_2, \ldots, a_{n-2})$$

for $n \geqslant 2$.)