9. EXPLICIT EXAMPLES OF TRANSCENDENTAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS

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in [0, 1] containing only 0's and 2's in their ternary expansion), then C has measure 0, and it is not hard to show that $C + C = [0, 2]$; see Borel [40] or Pavone [233]. The result is due to Steinhaus [310]; I am most grateful to G. Myerson for bringing this to my attention.

As we have seen above, the set $\mathscr{B}$, and hence each $\mathscr{B}_k$, also has Lebesgue measure zero. In 1947 Hall proved the following theorem [126]:

THEOREM 3. *Every real number* x *can be written as* $x = y + z$, *where* $y, z \in \mathscr{B}_4$. *Every real number* $x \geqslant 1$ *can be written as* $x = yz$, *where* $y, z \in \mathscr{B}_4$.

An exposition of Hall's result can be found in Cusick and Flahive [67].

Using the notation of the first paragraph of this section, we could rephrase the statement of Hall's theorem as follows: $\mathscr{B}_4 + \mathscr{B}_4 = \mathbf{R}$, and $[1, \infty) \subseteq \mathscr{B}_4 \cdot \mathscr{B}_4$.

In 1973, Cusick [61] proved that $\mathscr{B}_3 + \mathscr{B}_3 + \mathscr{B}_3 = \mathbf{R}$, and $\mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 = \mathbf{R}$. He also observed that $\mathscr{B}_3 + \mathscr{B}_3 \neq \mathbf{R}$, and $\mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 \neq \mathbf{R}$. These results were independently discovered by Diviš [90] and J. Hlavka[1]) [149]. Hlavka also showed that $\mathscr{B}_3 + \mathscr{B}_4 = \mathbf{R}$, and similar results. Apparently the status of $\mathscr{B}_2 + \mathscr{B}_5$ and $\mathscr{B}_2 + \mathscr{B}_6$ is still open.

For results of a similar character, see Cusick [60]; Cusick and Lee [68]; and Bumby [47].

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In Lang [179] we find the following statement:

> No simple example of [irrational] numbers of constant type, other than the one given above [real quadratic irrationals], is known. The best guess is that there are no other "natural" examples.

(Also see Lang [180].)

However, in 1979 Kmošek [167] and Shallit [275] independently discovered the following "natural" example of numbers of constant type.

THEOREM 4. *Let* $n \geqslant 2$ *be an integer and define*

(1) $$f(n) = \sum_{i \geqslant 0} n^{-2^i}.$$

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[1]) Note this is *not* same person as E. Hlawka!

Then $K(f(2)) = 6$ *and* $K(f(n)) = n + 2$ *for* $n \geqslant 3$.

For example, we have

$$f(3) = [0, 2, 5, 3, 3, 1, 3, 5, 3, 1, 5, 3, 1, \ldots] .$$

It is also possible to show that $K(nf(n)) = n$.

For related articles, see Köhler [171]; Pethö [237]; Shallit [277], and Wu [305]. (An aside: Mignotte [213] proved that there exists a constant c such that

$$\left| f(2) - \frac{p}{q} \right| > \frac{c}{q^3}$$

for all integers p and odd q. However, by combining Theorems 1 and 4, we get the improved bound

$$\left| f(2) - \frac{p}{q} \right| > \frac{1}{8q^2}$$

for all integers $q \geqslant 1$. Also see Derevyanko [86].)

Kempner [159] had proved in 1916 that $f(n)$ is transcendental for all integers $n \geqslant 2$. Mahler [200] also proved this result; also see Loxton and van der Poorten [195].

(Kempner seems to be responsible for a mistake that has been perpetuated in several papers. He called the series in Eq. (1) above the *Fredholm series,* in the belief that Fredholm studied it. Kempner referred to a paper of Mittag-Leffler [215], but this paper discusses the series

$$\sum_{i \geqslant 0} x^{i^2} ,$$

which is very different. An examination of Fredholm's collected works [108] did not turn up any papers on the series in Eq. (1). This mistaken attribution was repeated by Schneider in his classic work on transcendental numbers [273], and then repeated by other authors; see, e.g. Pethö [237]; Mendès France [207].)

Mendès France pointed out an intriguing connection between the continued fraction expansion of $f(n)$ and iterated paperfolding, which we now describe briefly.

If we fold a piece of paper in half repeatedly, say n times, always folding right hand over left hand, we get a series of $2^n - 1$ hills and valleys upon unfolding. Let us denote the hills by $+1$ and the valleys by -1. Letting X_n be the sequence of folds so obtained, it is not hard to see that

$$X_{n+1} = X_n \quad (+1) \quad -X_n^R ,$$

where juxtaposition denotes concatenation, and by X_n^R we mean the sequence X_n taken in reverse order.

More generally, we can choose to introduce a hill or valley at the nth fold. If we denote the nth fold by a_n, then after folding with $a_1, a_2, \ldots, a_n$, upon unfolding we get the sequence

$$F_{a_1}(F_{a_2}(\cdots(F_{a_n}(\varepsilon))\cdots)) ,$$

where ε denotes a sequence of length 0, and F_i is the *folding map,* given by

$$F_i(X) = X \quad i \quad -X^R .$$

Mendès France observed that the continued fraction expansion of $f(n)$ could be written in terms of the folding map F_i; see Mendès France [207]; Blanchard and Mendès France [33]; Dekking, van der Poorten and Mendès France [80]; Shallit [276]; and Mendès France and Shallit [209].

More recently, van der Poorten and Shallit [248] discovered a closer connection between paperfolding and continued fractions. Suppose we consider the formal power series

$$g(X) = \sum_{k \geqslant 0} X^{-2^k} \in \mathbf{Q}((1/X)) .$$

Then $Xg(X)$ can be expanded as a continued fraction, and it is not hard to prove that

$$Xg(X) = [1, F_{-X}(F_{-X}(\cdots(F_{-X}(X))\cdots))] ;$$

i.e. the continued fraction is given by the iterated folding of a piece of paper!

Using this result, we can prove the following theorem: let $\varepsilon_0 = 1$ and $\varepsilon_i = \pm 1$ for $i \geqslant 1$. Then the continued fraction expansion of each of the numbers

$$2 \sum_{i \geqslant 0} \varepsilon_i 2^{-2^i}$$

consists solely of 1's and 2's. For example,

$$2f(2) = [1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, \ldots] .$$

Let us now turn to other constructions of transcendental numbers with bounded partial quotients.

Since the set $\mathscr{B}$ is uncountable, while the set of algebraic numbers is countable, it is clear that almost all elements of $\mathscr{B}$ are transcendental. However, many investigators were concerned with the *explicit* construction of transcendental elements of $\mathscr{B}$. For example, Baker proved that

$$[0, 1, 2, 2, 1, 1, 1, 1, \overbrace{2, \ldots, 2}^{8}, \overbrace{1, \ldots, 1}^{16}, \overbrace{2, \ldots, 2}^{32}, \ldots]$$

and similar numbers are transcendental; see [16]. Previously, Maillet had given similar examples, but not explicitly [201]. Other examples have been recently given by Davison [79]. Also see Grant [120].

10. "Quasi-Monte-Carlo" Methods and Zaremba's Conjecture

In this section we briefly discuss some integration methods that depend on rational numbers with small partial quotients. There is a large literature on this subject; the interested reader can start with the comprehensive survey of Niederreiter [220].

(This section is tied to the main subject in the following manner: we wish to construct explicitly rational numbers with small partial quotients. One way to do this is to take an irrational number with bounded partial quotients and employ the sequence of convergents.)

In s-dimensional "quasi-Monte Carlo" integration, we approximate the integral

$$\int_{[0,1]^s} f(\mathbf{t})\,d\mathbf{t} \tag{2}$$

by the sum

$$\frac{1}{n} \sum_{1 \leqslant k \leqslant n} f(\mathbf{x}_k),$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is a set of points in $[0, 1]^s$.

The goal of quasi-Monte Carlo integration is to choose the points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ so as to minimize the error in the approximation.

In the method of *good lattice points,* we assume that the function f is periodic of period 1 in each variable. We choose a large fixed integer m and a special lattice point $\mathbf{g} \in \mathbf{Z}^s$. Then we approximate the integral (2) with the sum

$$\frac{1}{m} \sum_{1 \leqslant k \leqslant m} f\left(\frac{k}{m}\mathbf{g}\right).$$