14. PSEUDO-RANDOM NUMBER GENERATION

if the limit exists. Then Veech showed that $\mu_\theta(I)$ exists for all $I \subseteq [0, 1)$ if and only if the partial quotients of θ are bounded.

For other connections with ergodic theory, see the papers of Stewart [286]; del Junco [154]; Dani [70, 72]; and Baggett and Merrill [14, 15].

14. Pseudo-random Number Generation

Lehmer [183] introduced the *linear congruential method* for pseudo-random number generation. Let X_0, m, a, c be given, and define

$$X_{k+1} = aX_k + c \pmod{m},$$

for $k \geqslant 0$. For this to be a good source of "random" numbers, we want the sequence X_k to be uniformly distributed, as well as the sequence of pairs (X_k, X_{k+1}), triples, etc.

A test for randomness called the *serial test* on pairs (X_k, X_{k+1}) amounts to the two-dimensional version of the discrepancy mentioned above in Section 12. This turns out to be essentially the function $\rho(\mathbf{g}, m)$ defined in Section 10. Thus linear congruential generators that pass the pairwise serial test arise from rationals a/m having small partial quotients in their continued fraction expansion. See the papers of Dieter [87, 88]; Niederreiter [219, 220, 222]; Knuth [170, Section 3.3.3]; and Borosh and Niederreiter [42].

15. Formal Language Theory

Let $w = w_0 w_1 w_2 \cdots$ be an infinite word over a finite alphabet. We say that the finite word $x = x_0 x_1 \cdots x_n$ is a *subword* of w if there exists $m \geqslant 0$ such that $w_{m+i} = x_i$, for $0 \leqslant i \leqslant n$. We say that w is *k-th power free* if x^k is never a subword of w, for all nonempty words x. Here is a classical example: let $s(n)$ denote the number of 1's in the binary expansion of n. Then the infinite *word of Thue-Morse*

$$t = t_0 t_1 t_2 \cdots = 0110100110010110 \cdots,$$

defined by $t_n = s(n) \bmod 2$, is cube-free.

Another way to define infinite words is as the fixed point of a homomorphism on a finite alphabet. For example, the Thue-Morse word t is a fixed point of φ, where $\varphi(0) = 01$ and $\varphi(1) = 10$.