16. Other Results

A famous infinite word which has been extensively studied is the *Fibonacci word*

$$f = 101101011011010 \cdots ;$$

it is a fixed point of the homomorphism μ, where $\mu(1) = 10$ and $\mu(0) = 1$. For some of the properties of this word, see the survey of Berstel [28]. Karhumäki showed that f is fourth-power-free; see [155].

Now we define some special infinite words. Let $\theta \in [0, 1)$ and define the infinite word $w = w_1 w_2 w_3 \cdots$ as follows:

$$w_n = [(n+1)\theta] - [n\theta] .$$

If we set $\theta = (\sqrt{5} - 1)/2$, we get the Fibonacci word f. Recently, Mignosi [212] proved the following theorem: there exists a k such that w is k-th power-free, if and only if θ has bounded partial quotients. (One direction of Mignosi's theorem follows easily from two different descriptions of w in terms of the continued fraction expansion for θ; see Markoff [205]; Stolarsky [287]; and Fraenkel, Mushkin, and Tassa [107].)

16. Other Results

Let θ be an irrational number of constant type. Let p_n/q_n denote the n-th convergent to θ.

For n a positive integer, let $P(n)$ denote the largest prime factor of n. Then given $\varepsilon > 0$, there exists a constant $c = c(\theta; \varepsilon)$ such that the number of positive integers $n \leqslant x$ with

$$P(q_n) < c \log\log q_n$$

is at most εx. This is a result of Shorey [279].

Schmidt [269] showed that if $f_1, f_2, \ldots$ is a sequence of differentiable functions whose derivatives are continuous and vanish nowhere, then there are uncountably many numbers θ such that all the numbers $f_1(\theta), f_2(\theta), \ldots$ have bounded partial quotients. For related results, see Davenport [74, 75] and Cassels [51].

Other topics connected with real numbers with bounded partial quotients not discussed in this survey include transcendental number theory (see Baker [17]; Flicker [106]; Bundschuh [49]; Angell [11]), Fibonacci hashing on digital computers (see Knuth [169, pp. 510-512]), dynamical systems and global analysis (see Deligne [81]; Katznelson [156]; Herman [142, 143, 144, 145, 146]; Meyer [211]; de la Llave [193, 194]; MacKay [196, 197]; MacKay, Meiss, and

Percival [198]; Greene and MacKay [121]; Gutierrez [122]; Rand [255]; Stark [285]; Katznelson and Ornstein [157]; MacKay and Stark [199]; Sinai and Khanin [282]), and in the proof of a theorem connected with Kemperman's inequality (see Laczkovich [176]). For a connection with the "entropy" of a curve, see Mendès France [208].

17. Related Results

In this survey, we have restricted our attention to *real numbers* with bounded partial quotients. However, we would be remiss to omit mentioning the work on formal power series over a finite field having partial quotients of bounded degree. See the papers of Baum and Sweet [23, 24]; Mills and Robbins [214]; Mesirov and Sweet [210]; Mullen and Niederreiter [216]; Niederreiter [225, 227, 226]; Allouche [8]; and Allouche, Mendès France, and van der Poorten [10].

It is perhaps appropriate to mention the following question of Mendès France, which asks (roughly) if a formal power series over a finite field is algebraic and the partial quotients in its continued fraction expansion are of bounded degree, then must those partial quotients be accepted by a finite automaton? For a more precise version of this conjecture, see the paper of Allouche, Betrema, and Shallit [9]. This paper also gives some examples for which the answer to Mendès France's question is positive. However, the partial quotients in the continued fraction for the power series of Baum and Sweet [23], which were later described explicitly by Mills and Robbins [214], do not seem to be accepted by a finite automaton.

18. Acknowledgments

The author would like to express his thanks to the library staff at Darthmout College and the University of Waterloo, for their invaluable assistance in locating some of the more obscure references presented here.

Thanks to A. Baker, J. L. Davison, T. Cusick, K. Dilcher, H. Niederreiter, A. J. van der Poorten, M. Mendès France, R. Bumby, D. Yang, and A. Pethö for providing pointers to the literature. I. Vardi and J. C. Lagarias read an early version of this paper and made many useful suggestions. A. Bultheel and B. Swartz provided a reference to the work of Cotes. I am most grateful to J. Wolfskill, who pointed out that the results in Davenport and Roth [77] could be made effective.