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# ELLIPTIC SPACES II

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ABSTRACT. A simply connected finite $CW$ complex $X$ is *elliptic* if the homology of its loop space (coefficients in any field) grows at most polynomially. We show that in all other cases the loop space homology grows at least semi-exponentially, and we exhibit a number of geometrically interesting classes of spaces as elliptic, including: $H$ spaces, homogeneous spaces, Poincaré duality complexes whose mod $p$ cohomology is doubly generated (any $p$) and Dupin hypersurfaces in $S^{n+1}$.

## 1. INTRODUCTION

Let $X$ be a simply connected finite $CW$ complex, with loop space $\Omega X$, and denote by $\mathbf{F}_p$, the prime field of characteristic $p$, $p$ prime or zero. Our first main result asserts a dichotomy for the size of the loop space homology $H_*(\Omega X; \mathbf{F}_p)$:

THEOREM A. *Let $X$ be a simply connected finite $CW$ complex. For each $p$ (prime or zero) there are exactly two possibilities: either*

(i) *There are constants $C > 0$ and $r \in \mathbf{N}$ such that*

$$\sum_{i=0}^{n} \dim H_i(\Omega X; \mathbf{F}_p) \leqslant Cn^r, \qquad n \geqslant 1\ ,$$

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*Key words:* loop space homology, depth, polynomial growth, Poincaré complex, elliptic, Dupin hypersurface.

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*or else*

(ii) *There are constants* $K > 1$ *and* $N \in \mathbf{N}$ *such that*

$$\sum_{i=0}^{n} \dim H_i(\Omega X; \mathbf{F}_p) \geqslant K^{\sqrt{n}}, \quad n \geqslant N.$$

In case (i) the loop space homology grows *at most polynomially,* and $X$ is $\mathbf{Z}_{(p)}$-elliptic in the sense of [6]. If (i) holds for all $p$ then $X$ is elliptic. The main theorems of [6] assert that if $X$ is elliptic then $X$ is a Poincaré complex and that $H_*(\Omega X; \mathbf{Z})$ is a finitely generated left noetherian ring.

In case (ii) above the loop space homology grows *at least semi-exponentially*. However, when $p = 0$ [2] or $p \geqslant \dim X$ [8], it can be shown that even the primitive subspace of $H_*(\Omega X; \mathbf{F}_p)$ grows exponentially (implying the same result for $H_*(\Omega X; \mathbf{F}_p)$), and we conjecture that this should hold true for all $p$.

In the dichotomy of Theorem A, the generic situation is (ii): elliptic spaces are rare within the class of all simply connected finite $CW$ complexes. However a number of geometrically interesting spaces are elliptic, and our second objective in this note is to show that these include the following classes of spaces (provided they are simply connected):

finite $H$-spaces,

homogeneous spaces,

spaces admitting a fibration $F \to X \to B$ with $F$, $B$ elliptic,

Poincaré complexes $X$ such that for each $p$, the algebra $H^*(X; \mathbf{F}_p)$ is generated by two elements,

Dupin hypersurfaces in $S^{n+1}$,

closed manifolds admitting a smooth action by a compact Lie group, with a simply connected codimension one orbit,

connected sums $M \# N$ with the algebras $H^*(M; \mathbf{Z})$ and $H^*(N; \mathbf{Z})$ each generated by a single class.

This note is sequel to "Elliptic Spaces" [6]. In particular, it supersedes the preprint "Dupin hypersurfaces are elliptic" referred to in [6].

## 2. The dichotomy

Consider first any simply connected space $X$ with each $H_i(X; \mathbf{F}_p)$ finite dimensional. Then $G = H_*(\Omega X; \mathbf{F}_p)$ is a graded cocommutative Hopf algebra satisfying $G_0 = \mathbf{F}_p$ and each $G_i$ is finite dimensional. The *depth* of $G$