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INTRODUCTION

Let $f: (\mathbf{X}, x) \to (\mathbf{C}, 0)$ be a holomorphic function on an analytic germ $(\mathbf{X}, x)$. Let h_f denote the monodromy of the germ $\Psi_f^\bullet(\mathbf{C}_\mathbf{X}^\bullet)_x$ of neighbouring cycles. One defines its *Lefschetz number*

$$\Lambda(h_f) := \sum_{i \geqslant 0} (-1)^i \text{ trace } [h_f; \Psi_f^i(\mathbf{C}_\mathbf{X}^\bullet)_x] ,$$

and its *zeta-function*

$$\zeta_{h_f}(t) := \prod_{i \geqslant 0} \det[I - t \cdot h_f ; \Psi_f^i(\mathbf{C}_\mathbf{X}^\bullet)_x]^{(-1)^{i+1}} .$$

We alternatively denote them by $\Lambda(f)$, respectively $\zeta_f(t)$.

A theorem of Eisenbud and Neumann [EN, Theorem 4.3] asserts that the zeta-function of a *fibred multilink* L is the product of the zeta-functions over all *splice components* of L. If the multilink is defined by some Cerf diagram $\Delta(l, f)$, then $\zeta_f(t)$ becomes the zeta-function of the multilink L, this time with coefficients in a local system. This observation of Némethi [Ne] enables him to prove an inductive formula for $\zeta_f(t)$, in terms of invariants of the so called EN-diagram (splice diagram); compare to the one of Eisenbud and Neumann [EN, p. 96]. Some quite strong results in the 3-dimensional link theory are involved in the proofs.

Our approach is based on Lê's carrousel construction and is therefore more geometric and selfcontained. It yields inductive formulae for $\Lambda(f)$ and $\zeta_f(t)$ directly from the Puiseux parametrization of $\Delta(l, f)$. Moreover, it clarifies the contribution, however essential in general, of the "nonessential" terms in this parametrization — which may be not clear from the definition of the splice diagram of an algebraic link given in [EN, p. 53], simply because such terms are completely omitted. One can therefore compare to our definitions 1.5 ÷ 7.

The formula for $\zeta_f(t)$ will be not the same, but quite similar to the ones before. The ingredients are zeta-functions of fibres over certain periodic points in the carrousel disc. We show in Sections 2 and 3 how to define these points from the Puiseux expansion of $\Delta(l, f)$. We end by some applications.

Aknowledgement. This work is based on a piece of the author's disertation [Ti]. He much benefited from talks with Dirk Siersma, whose paper [Si] incited him to do this research (see 3.8).

1. THE CARROUSEL REVISITED

1.1. We first briefly recall the carrousel construction, following closely [Lê-1] and [Lê-3], then give the necessary definitions for our study. One regards $(\mathbf{X}, x)$ as being embedded in $(\mathbf{C}^N, 0)$, for some sufficiently large $N \in \mathbf{N}$. We assume that, unless otherwise stated, all the irreducible components of $(\mathbf{X}, 0)$ have dimensions greater than 1.

Let $\mathscr{X}$ be a small enough representative of $(\mathbf{X}, 0)$. Let $\Gamma(l, f)$ be the *polar curve* of f with respect to a linear function $l: (\mathbf{X}, 0) \to (\mathbf{C}, 0)$, relatively to a fixed *Whitney stratification* $\mathscr{S}$ on $\mathscr{X}$ which satisfies *Thom condition* (a_f).

The polar curve $\Gamma(l, f)$ exists for a Zariski open subset $\hat{\Omega}_f$ in the space of linear germs $l: (\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$. If one does not impose $\Gamma(l, f)$ to be reduced then one gets a larger set $\Omega_f \supset \hat{\Omega}_f$ which is sometimes useful to deal with (see e.g. Example 2.2). (We only mention that one can enlarge even Ω_f: modify its definition by allowing also nonlinear functions.)

1.2. Let $l \in \Omega_f$ and let $\Phi := (l, f): (\mathbf{X}, 0) \to (\mathbf{C}^2, 0)$. We denote by (u, λ) the pair of coordinates on $\mathbf{C}^2$.

The curve germ (with reduced structure) $\Delta(l, f) := \Phi(\Gamma(l, f))$ is called the *Cerf diagram* (of f with respect to l, relative to $\mathscr{S}$). We shall use the same notation $\Gamma(l, f)$, respectively $\Delta(l, f)$ for suitable representatives of these germs.

There is a fundamental system of "privileged" open polydiscs in $\mathbf{C}^N$, centred at 0, of the form $(D_\alpha \times P_\alpha)_{\alpha \in A}$ and a corresponding fundamental system $(D_\alpha \times D'_\alpha)_{\alpha \in A}$ of 2-discs at 0 in $\mathbf{C}^2$, such that Φ induces, for any $\alpha \in A$, a topological fibration

$$\Phi_\alpha: \mathscr{X} \cap (D_\alpha \times P_\alpha) \cap \Phi^{-1}(D_\alpha \times D'_\alpha \setminus (\Delta(l, f) \cup \{\lambda = 0\})) \to D_\alpha \times D'_\alpha \setminus (\Delta(l, f) \cup \{\lambda = 0\}) .$$

Moreover, f induces a topological fibration

$$f_\alpha : \mathscr{X} \cap (D_\alpha \times P_\alpha) \cap f^{-1}(D'_\alpha \setminus \{0\}) \to D'_\alpha \setminus \{0\}\,,$$

respectively

$$f'_\alpha : \mathscr{X} \cap (\{0\} \times P_\alpha) \cap f^{-1}(D'_\alpha \setminus \{0\}) \to D'_\alpha \setminus \{0\}\,,$$

which is fibre homeomorphic to the Milnor fibration of f, respectively to the Milnor fibration of $f_{|\{l=0\}}$. The disc D'_α has been chosen small enough such that $\Delta(l,f) \cap \partial \overline{D_\alpha} \times D'_\alpha = \varnothing$.

1.3. One can build an integrable smooth vector field on $D_\alpha \times S'_\alpha$ — where S'_α is some circle in D'_α of radius sufficiently close to the radius of $\partial \overline{D'_\alpha}$ — such that, mainly, it is tangent to $\Delta(l,f) \cap (D_\alpha \times S'_\alpha)$ and it lifts the unit vector field of S'_α by the projection $D_\alpha \times S'_\alpha \to S'_\alpha$. Lifting the former vector field by Φ_α and integrating it, one gets a characteristic homeomorphism of the fibration induced by f_α over S'_α, hence a geometric monodromy of the Milnor fibre F_f of f. We call it the (geometric) *carrousel monodromy.*

For some fixed $\eta \in S'_\alpha$, let

(1) $$l_\alpha : \mathscr{X} \cap \Phi_\alpha^{-1}(D_\alpha \times \{\eta\}) \to D_\alpha \times \{\eta\}$$

be the restriction of Φ_α and notice that F_f is homeomorphic to $l_\alpha^{-1}(D_\alpha \times \{\eta\})$.

The integration of the vector field on $D_\alpha \times S'_\alpha$ produces a "carrousel" of the disc $D_\alpha \times \{\eta\}$: the trajectory inside $D_\alpha \times S'_\alpha$ of some point $a \in D_\alpha \times \{\eta\}$ projects onto S'_α; one turn around the circle S'_α moves the point a to some other point $a' \in D_\alpha \times \{\eta\}$. By construction, the vector field restricted to $\{0\} \times S'_\alpha$ is the unit vector field of S'_α, hence the centre $(0, \eta)$ of the carroussel disc is indeed fixed; the circle $\partial \overline{D_\alpha} \times \{\eta\}$ is also pointwise fixed.

The distinguished points $\Delta(l,f) \cap D_\alpha \times \{\eta\}$ of the disc have a complex motion around $(0, \eta)$, depending on the Puiseux parametrizations of the branches of Δ which are not included in $\{u = 0\}$. Moreover, these Puiseux expansions determine the motion of any "important" point in the carrousel, as briefly described in the next.

1.4. Our notation is close to the one in [BK].

Let $\Delta := \Delta(l,f)$ and let $\Delta' = \cup_{i \in \{1, \dots, r\}} \Delta_i$ be the union of those irreducible components of Δ which are not included in $\{u = 0\}$.

For $i \in \{1, \ldots, r\}$, we consider a Puiseux parametrization of Δ_i with reduced structure:

$$(2)\qquad \begin{cases} \lambda = t^n \\ u = \sum_{j \geqslant m} c_j t^j, \quad \text{for some } m, n \in \mathbf{Z}_+, c_j \in \mathbf{C}, c_m \neq 0. \end{cases}$$

Notice that $m \leqslant n$. The Puiseux parametrization enables one to formally write u as a function of λ:

$$(3)\qquad \begin{aligned} u = a_{k_1}\lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l}\lambda^{(m_1+l)/n_1} + a_{k_2}\lambda^{m_2/n_1n_2} \\ + \sum_{l=1}^{l_2} b_{2,l}\lambda^{(m_2+l)/n_1n_2} + \cdots + a_{k_g}\lambda^{m_g/n_1\cdots n_g} + \sum_{l>0} b_{g,l}\lambda^{(m_g+l)/n_1\cdots n_g}, \end{aligned}$$

where g is a positive integer, $\gcd(m_j, n_j) = 1, \forall j \in \{1, \ldots,, g\}$ and $n_j \neq 1$, $\forall j \in \{2, \ldots, g\}$. Notice that $m_1/n_1 = m/n$ and $a_{k_1} = c_m$.

1.5. We now define two sequences $\{C_i^{(j)}\}_{j \in \{1, \ldots g\}}$, $\{\hat{C}_i^{(j)}\}_{j \in \{1, \ldots g\}}$ of successive approximation of Δ_i, $i \in \{1, \ldots, r\}$:

$$C_i^{(j)}: u = a_{k_1}\lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l}\lambda^{(m_1+l)/n_1} + \cdots + a_{k_j}\lambda^{m_j/n_1\cdots n_j},$$

$$\hat{C}_i^{(j)}: u = a_{k_1}\lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l}\lambda^{(m_1+l)/n_1} + \cdots + a_{k_j}\lambda^{m_j/n_1\cdots n_j} + \sum_{l=1}^{l_j} b_{j,l}\lambda^{(m_j+l)/n_1\cdots n_j}$$

and $\hat{C}_i^{(g)} = \Delta_i$.

The curve $C_i^{(1)}$ intersects the carrousel disc $D_\alpha \times \{\eta\}$ in n_1 points situated on a circle and their carrousel motion is a rotation of angle $2\pi m_1/n_1$. If we take $\hat{C}_i^{(1)}$ instead, we get also n_1 intersection points but their position is a slight perturbation of the previous one.

Each of the points $C_i^{(1)} \cap (D_\alpha \times \{\eta\})$ is the centre of a small disc which contains just one point from the set $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$. This latter one, called a *distinguished point,* becomes the centre of a *new (smaller) carrousel.*

Our next definition will play a central role.

1.6. *Definition.* We call *carrousel disc of order k* a sufficiently small open disc centred at some point $c \in \hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$, $i \in \{1, \ldots, r\}$, which contains all the points $\hat{C}_j^{(k+l)} \cap (D_\alpha \times \{\eta\}), \forall l > 0, \forall j \in \{1, \ldots, r\}$ such that $\hat{C}_j^{(k)} = \hat{C}_i^{(k)}$, which are close enough ("satellites") to c. If δ_1, δ_2 are two smaller carrousel discs (not necessarily of the same order), then they are either disjoint or included one in the other.

We may and do assume that the carrousel discs of order k centred at the points $\hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$, $i \in \{1, \ldots, r\}$, are of equal radii.

Remark. A small carrousel disc of order k may contain other carrousel discs of the same order. In the next example:

$$\Delta_1: \ u_1 = \lambda^{3/2} + \lambda^{17/2}, \quad C_1^{(1)} \neq \hat{C}_1^{(1)}, \ \Delta_1 = \hat{C}_1^{(1)},$$
$$\Delta_2: \ u_2 = \lambda^{3/2} + \lambda^{7/4}, \quad C_2^{(1)} = \hat{C}_2^{(1)} = C_1^{(1)}, \ \Delta_2 = C_2^{(2)},$$

a carrousel disc of order 1 corresponding to Δ_2 contains a carrousel disc of order 1 corresponding to Δ_1.

1.7. Finally, a simultaneous parametrization of all analytic branches of $\Delta' : \lambda = t^n$, $u_k = \Sigma_{j \geqslant m_k} a_{k,j} t^j$, for $k \in \{1, \ldots, r\}$, leads to the construction of the full carrousel.

If we define the "essential" curve associated to Δ_i by:

$$\Delta_i^{\text{es}}: \ u = a_{k_1} \lambda^{m_1/n_1} + a_{k_2} \lambda^{m_2/n_1 n_2} + \cdots + a_{k_g} \lambda^{m_g/n_1 \cdots n_g},$$

then the carrousel associated to $\Delta^{\text{es}} = \bigcup_{i \in \{1, \ldots, r\}} \Delta_i^{\text{es}}$ might be called an "ideal carrousel". However, the topological type of the link Δ' may be *not* the same as the one of Δ^{es}.

1.8. Denote by $(m_{i,j}, n_{i,j})_{j \in \{1, \ldots, g_i\}}$ the Puiseux pairs of Δ_i, $\forall i \in \{1, \ldots, r\}$. Suppose that we have the following ordering among the first Puiseux pairs (eventually after some permutation of indices): $m_{1,1}/n_{1,1} \geqslant m_{2,1}/n_{2,1} \geqslant \cdots \geqslant m_{r,1}/n_{r,1}$.

To each branch Δ_i there corresponds an annulus A_i — with central symmetry at $(0, \eta)$ — inside the carrousel disc, such that A_i contains $\Delta_i \cap (D_\alpha \times \{\eta\})$, see [Lê-1]. We also define A_0 to be an arbitrarily small open disc centred in $(0, \eta)$. By definition, $A_i = A_j$ if and only if $m_{i,1}/n_{i,1} = m_{j,1}/n_{j,1}$.

For any $i \in \{1, \ldots, r\}$, there are $n_{i,1}$ carrousel discs $\delta_{i,j}$, $j \in \{1, \ldots, n_{i,1}\}$, of order 1, centred at the $n_{i,1}$ points $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$. In case of the "ideal" carrousel, these points rotate around $(0, \eta)$ by $2\pi m_{i,1}/n_{i,1}$. The annulus A_i contains all the carrousel discs $\delta_{s,j}$ such that $C_s^{(1)} = C_i^{(1)}$. Each point of the annulus A_i, outside any disc $\delta_{s,j}$, is fixed by the $n_{i,1}$th iterate of the carrousel. The disc A_0 is just pointwise fixed by the carrousel.

Of course, one needs a continuous transition between two annuli. The *transition zone* will be a sufficiently thin annulus connecting A_i to A_{i+1}, such that the collection of A_i's and transition zones give a partition of the carrousel disc.

2. Lefschetz number via the carrousel

Let $\mathbf{m}_{\mathbf{X},0}$ denote the maximal ideal of the local ring $\mathcal{O}_{\mathbf{X},0}$. A'Campo proves via the resolution of singularities that, if $f \in \mathbf{m}^2_{\mathbf{X},0}$, then $\Lambda(f) = 0$ ([A'C-1, Théorème 1bis]).

Alternatively, the carrousel construction can provide information on the Lefschetz number. This was the idea of Lê, who showed that, if $f \in \mathbf{m}^2_{\mathbf{X},0}$, and $(\mathbf{X}, 0)$ is smooth, then the carrousel monodromy has no fixed points outside the slice $\{l = 0\}$, so $\Lambda(f) = 0$ by induction.

We extend this result by studying the set of fixed points in case $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$.

2.1. Theorem. *Let all the irreducible components of* $(\mathbf{X}, 0)$ *have dimensions greater than* 1. *If* $n_{i,1} > 1$, $\forall i \in \{1, \dots, r\}$, *then* $\Lambda(f) = \Lambda(f_{|\{l=0\}})$.

Proof. Assume that $\Delta \not\subset \{u = 0\}$. Since $n_{i,1} > 1$, the carrousel construction tells us that the discs $\delta_{s,j}$ (defined in 1.8), with $n_{s,1} = n_{i,1}$, are cyclically permuted (by a cycle of length $n_{i,1}$).

We may conclude that no point in the carrousel disc is fixed, except the centre and, possibly, some subsets in the transition zones. In each transition zone the subset of fixed points is a finite union of circles, all centred at $(0, \eta)$.

One can decompose the Milnor fibre F_f into suitable pieces on which the geometric monodromy acts and such that the Mayer-Vietoris exact sequence can be applied. Actually, we first cover the carrousel disc by some annuli like those defined in 1.8, then lift this patching to the Milnor fibre. If A_0 is small enough, then $l_\alpha^{-1}(0, \eta)$ is a deformation retract of $l_\alpha^{-1}(A_0)$.

We may conclude: $\Lambda(f) = \Lambda(f_{|\{l=0\}})$, provided that the Lefschetz number of the restriction of the monodromy on any piece of F_f which is the lift by l_α of some pointwise fixed circle is zero. This fact is emphasized in the next lemma, whose proof is left to the reader. The case $\Delta \subset \{u = 0\}$ leads to the same conclusion. □

Lemma. *If the carrousel disc* $D_\alpha \times \{\eta\}$ *contains a circle* S *of fixed points, all of them regular values for the map* l_α, *then* $\Lambda(h_f; H^\bullet(l_\alpha^{-1}(S))) = 0$. □

2.2. *Example.* Let $(\mathbf{X}, 0)$ be a 2-dimensional isolated cyclic quotient singularity, where $\mathbf{X}$ is the algebraic quotient of $\mathbf{C}^2$ by a cyclic group of order 5, usually denoted by $\mathbf{X}_{5,2}$: if ξ is a primitive 5-root of 1, then a generator of our group acts on $\mathbf{C}^2$ by $(x, y) \mapsto (\xi x, \xi^2 y)$.

Let $\tilde{f}: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$, $\tilde{f} = x^5 + y^5$ and let $f: (\mathbf{X}, 0) \to (\mathbf{C}, 0)$ be the induced function on the quotient. Take a function $\tilde{l}: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$, $\tilde{l} = xy^2$ and let l be the induced linear function on $(\mathbf{X}, 0)$. Then $l \notin \hat{\Omega}_f$, but $l \in \Omega_f$. Notice that $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$.

We get that $\Delta'(l, f)$ is irreducible and has a 1-term Puiseux parametrization with Puiseux pair $(3, 5)$. There follows $\Lambda(f) = \Lambda(f_{|\{l=0\}})$.

The Milnor fibre of $f_{|\{l=0\}}$ has two components: each of them is the Milnor fibre of a linear function on $(\mathbf{C}, 0)$. This implies that $\Lambda(f_{|\{l=0\}}) = 2$, hence $\Lambda(f) = 2$.

2.3. Corollary [A'C-1, Théorème 1bis]. *Let* $(\mathbf{X}, 0)$ *be an analytic germ of dimension* $\geqslant 1$. *If* $f \in \mathbf{m}^2_{\mathbf{X},0}$ *then* $\Lambda(f) = 0$.

Proof. Let $(\mathbf{X}, 0) = (\mathbf{X}_1, 0) \cup (\mathbf{X}_2, 0)$, where $(\mathbf{X}_2, 0)$ is the union of the irreducible components of $(\mathbf{X}, 0)$ which are of dimension $\geqslant 2$ and $(\mathbf{X}_1, 0)$ is the union of the 1-dimensional branches of $(\mathbf{X}, 0)$.

We slice $(\mathbf{X}_2, 0)$ by a general hyperplane defined by some $l \in \Omega_f$ and treat separately the 1-dimensional components of the slice. If $f \in \mathbf{m}^2_{\mathbf{X}_2,0}$ then each component of the Cerf diagram $\Delta(l, f)$ is tangent to the axis $\{\lambda = 0\}$, provided that l is *general enough*. The proof of this fact is similar to the proof of [Lê-4, Proposition 1.2], but this time the underlying space may be not smooth (see [Ti] for details).

Tangency to $\{\lambda = 0\}$ means exactly that $m_{i,1}/n_{i,1} < 1$, in particular $n_{i,1} > 1$, $\forall i \in \{1, \ldots, r\}$. Thus, our proof relays on a decreasing induction: at each step, we may apply Theorem 2.1. The assertion for 1-dimensional branches is proved by the next easy lemma. □

Lemma. *If* $(\mathbf{X}, 0)$ *is 1-dimensional, irreducible and if* $f \in \mathbf{m}^2_{\mathbf{X},0}$ *then there is a geometric monodromy of* f *without fixed points.* □

As a complement to Theorem 2.1, we have the following precise determination of the Lefschetz number in case $\dim(\mathbf{X}, 0) = 1$:

2.4. Proposition. *If* $(\mathbf{X}, 0) = \bigcup_{i \in R}(C_i, 0)$ *is a curve and its decomposition into irreducible components, then, for any* $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$, *we have:*

$$\Lambda(f) = \#\{i \in R \mid (C_i, 0) \text{ is smooth and } f \in \mathbf{m}_{C_i,0} \setminus \mathbf{m}^2_{C_i,0}\}.$$

Proof. Let $f_i := f_{|(C_i, 0)}$. Then the Milnor fibre of f is a finite set, the disjoint union of the Milnor fibres of f_i, $i \in R$. Hence, $\Lambda(f) = \Sigma_{i \in R}\Lambda(f_i)$.

If $(C_i, 0)$ is smooth, then one has: $\Lambda(f_i) = 1$ if and only if $f_i \in \mathbf{m}_{C_i, 0} \setminus \mathbf{m}^2_{C_i, 0}$.

If $(C_i, 0)$ is not smooth, let $n_i: (\tilde{C}_i, a_i) \to (C_i, 0)$ be its normalization. It follows $f_i \circ n_i \in \mathbf{m}^2_{\tilde{C}_i, a_i}$, hence the geometric monodromy of f_i is fixed-point-free and $\Lambda(f_i) = 0$. □

2.5. Define $P^{(1)} := \{i \in \{1, \ldots, r\} \mid n_{i,1} = 1\}$.

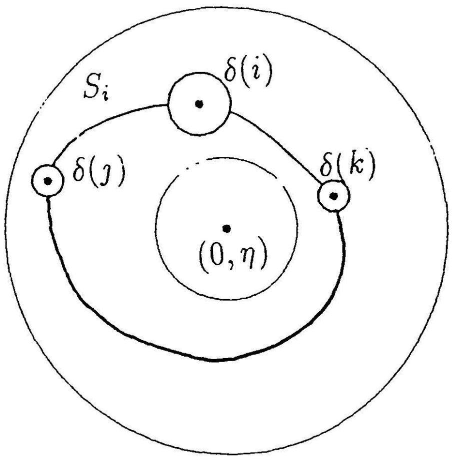
For $i \in P^{(1)}$, let B_i be the union of all carrousel discs of order 1 included in A_i. Then the carrousel construction tells us that the set $A_i \setminus B_i$ is pointwise fixed.

Further, let $\delta(j) \subset A_i$ be a carrousel disc of order 1 defined in the next 2.6. If there are no carrousel discs of order 1 included in $\delta(i)$, then the only fixed point of $\delta(i)$ is its centre. If $\delta(i)$ contains some carrousel disc of order 1 (see Remark 1.6), then we decompose $\delta(i)$ into annuli, since $\delta(i)$ is itself a carrousel. For those annuli that contain some carrousel disc of order 1, we may adapt the present argument, from the beginning of 2.5.

It is easily seen that the set $A_i \setminus B_i$, for $i \in P^{(1)}$, retracts to the subset:

$$(S_i \setminus \bigcup_{\delta \in \mathcal{K}_i^{(1)}} \delta) \cup \bigcup_{\delta \in \mathcal{K}_i^{(1)}} \partial\bar{\delta}, \tag{4}$$

where $\mathcal{K}_i^{(1)}$ is the set of carrousel discs of order 1 in A_i which are not included in other carrousel discs of the same order and S_i is a closed curve homotopic to a circle which intersects all the discs $\delta \in \mathcal{K}_i^{(1)}$.



The picture shows a possible shape of the retract of the set of fixed points inside $A_i \setminus B_i$: the "thick" curves are fixed. (The situation in the picture corresponds to $n_{i,1}/m_{i,1} = n_{j,1}/m_{j,1} = n_{k,1}/m_{k,1}$).

Then some neighbourhood of the set of fixed points after one turn of the big carrousel retracts to a set with a finite number of connected components, each of which being either:

(a) a circle centred at $(0, \eta)$ or at a centre of some carrousel disc of order 1, or

(b) a set defined as in (4) or — if case — a similar one in a carrousel disc of order 1, or

(c) a centre of a carrousel disc of order 1 inside A_i, for some $i \in P^{(1)}$, or

(d) the centre $(0, \eta)$ of the big carrousel disc.

2.6. *Definition.* Let $\mathscr{I}^{(1)}$ be a maximal set of indices $i \in P^{(1)}$ such that, if $i_1, i_2 \in \mathscr{I}^{(1)}$, then $\hat{C}^{(1)}_{i_1} \neq \hat{C}^{(1)}_{i_2}$.

For any $i \in \mathscr{I}^{(1)}$, denote by $\delta(i)$ the carrousel disc of order 1 centred at the point $c(i) := \hat{C}^{(1)}_i \cap (D_\alpha \times \{\eta\})$. Let $a(i)$ be an arbitrarily chosen point on the boundary $\partial\bar{\delta}(i)$; it is, by definition, a regular value for l_α.

Definition. Let $a \in (D_\alpha \backslash 0) \times \{\eta\}$ and let F'_a be the fibre of l_α over a. If a is fixed by the carrousel, then the monodromy h_f restricts to an action on $H^\bullet(F'_a)$, denoted by h'_a.

With these notations, we may formulate the following:

2.7. THEOREM. *If* $f \in \mathbf{m}_{X,0}$ *and* $l \in \Omega_f$, *then:*

$$\Lambda(f) = \Lambda(f_{|\{l=0\}}) + \sum_{i \in \mathscr{I}^{(1)}} [\Lambda(h'_{c(i)}) - \Lambda(h'_{a(i)})] .$$

Proof. The Lefschetz number $\Lambda(f)$ splits into a sum, following the decomposition of the set of fixed points into connected components, see 2.5. We use a suitable open covering of a set defined as in (4) and then apply the Mayer-Vietoris exact sequence. The reason of considering the set $\mathscr{I}^{(1)}$ relies on the above discution. By a straightforward computation, using also Lemma 2.1, we get our formula. □

Notice that carrousel discs of order $\geqslant 2$ do not enter in the above formula. For the computation of $\Lambda(h'_{c(i)})$, $\Lambda(h'_{a(i)})$, we refer to Remarks 3.6. There will be an example at the end.

3. ZETA-FUNCTION AND CARROUSEL MONODROMIES

3.1. Loosely speaking, each "important point" of the carrousel disc is fixed after a finite number of turns of the carrousel. We have seen that the set of fixed points after one turn determines the Lefschetz number $\Lambda(h_f)$. So the set of fixed points after k turns is the one responsible for the number $\Lambda(h^k_f)$. It may contain a finite number of circles consisting of regular values for the map l_α. Actually, these circles do not count, as shown by Lemma 2.1 (where

h_f has to be replaced by h_f^k). By examining the proof of Theorem 2.1, we get a slightly more general result:

PROPOSITION. *Let* $k \geqslant 1$. *If* $n_{i,1} \nmid k$, $\forall i \in \{1, \ldots, r\}$, *then* $\Lambda(h_f^k) = \Lambda(h_{f|\{l=0\}}^k)$. □

3.2. *Definition.* Let $U \subset D_\alpha \times \{\eta\}$ and let $k_U := \min\{k \mid U$ is globally fixed by the k^{th} iteration of the carrousel$\}$. Then $k_f^{k_U}$ restricts to an action on $H^\bullet(l_\alpha^{-1}(U))$, which we denote by h'_U. We call such actions *carrousel monodromies.*

3.3. The zeta-function is determined by the set of Lefschetz numbers $\Lambda(h_f^k)$, $k \geqslant 1$, as follows (see e.g. [Mi, p. 77], [A'C-2, p. 234]). If the integers $s_1, s_2, \ldots$ are inductively defined by $\Lambda(h_f^k) = \Sigma_{i|k} s_i$, $k \geqslant 1$, then the zeta-function of f is given by:

$$\xi_f(t) = \prod_{i \geqslant 1} (1 - t^i)^{-s_i/i} . \tag{5}$$

On the other hand, if $\mathscr{B}^{(k)}$ denotes some small enough neighbourhood of the set of fixed points of the k^{th} power of the carrousel, then h_f^k acts on the cohomology $H^\bullet(l_\alpha^{-1}(\mathscr{B}^{(k)}))$ and, with the definition above, we get $\Lambda(h_f^k) = \Lambda(h'_{\mathscr{B}^{(k)}})$.

Let's consider the annulus A_i, as before, in the big carrousel disc. Denote by h_{A_i} the restriction of h_f to $H^\bullet(l_\alpha^{-1}(A_i))$.

If $x \in A_i$ is fixed by some power k of the carrousel, then this power has to be a multiple of $n_{i,1}$. This remark and formula (5) yield the relation:

$$[\zeta_{h_{A_i}}(t)]^{n_{i,1}} = \zeta_{h_{A_i}^{n_{i,1}}}(t^{n_{i,1}}) . \tag{6}$$

Definition. For any $i \in \{1, \ldots, r\}$, denote by $\delta(i)^{(1)}$ the carrousel disc of order 1 centred at an arbitrarily chosen point of $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$, but fixed once and for all.

Let $\mathscr{L}^{(1)} := \{\delta = \delta(i)^{(1)} \mid i \in \{1, \ldots, r\}$, $\delta(i)^{(1)}$ is not contained in any other carrousel disc of order 1$\}$. For $\delta \in \mathscr{L}^{(1)}$, denote by $a(\delta)$ an arbitrarily chosen point on the boundary $\partial\bar{\delta}$.

Then we have the next recursive formula:

3.4. THEOREM. $\zeta_f(t) = \zeta_{f|\{l=0\}}(t) \cdot \prod_{\delta \in \mathscr{L}^{(1)}} \zeta_{h'_\delta}(t^{n_{i,1}}) \cdot \zeta_{h'_{a(\delta)}}^{-1}(t^{n_{i,1}})$.

Proof. We apply Mayer-Vietoris exact sequences to the covering of the carrousel disc described before. Since the fixed circles do not count for the

Lefschetz numbers, we get that the zeta-function is a product over all different annuli, each factor being of the form $\zeta_{h_{A_i}}(t)$.

We employ the notations in 2.5. Notice that the set $\mathscr{K}_i^{(1)}$ is well defined for any $i \in \{1, \ldots, r\}$. One can easily show that A_i retracts to the subset $\mathscr{R}_i := S_i \cup \bigcup_{\delta \in \mathscr{K}_i^{(1)}} \delta$, hence $\zeta_{h_{A_i}^{n_{i,1}}}(t) = \zeta_{h'_{\mathscr{R}_i}}(t)$.

If $\delta \in \mathscr{K}_i^{(1)}$, then notice that there are $n_{i,1}$ carrousel discs in A_i of the same radius as δ; if δ_1, δ_2 are any two of them, then $\zeta_{h'_{\delta_1}}(t) = \zeta_{h'_{\delta_2}}(t)$.

An open covering of $\mathscr{R}_i$ and a Mayer-Vietoris argument lead to the conclusion:

$$\zeta_{h'_{\mathscr{R}_i}}(t) = \prod_{\delta \in \mathscr{L}^{(1)}} [\zeta_{h'_\delta}(t)]^{n_{i,1}} \cdot [\zeta_{h'_{a(\delta)}}(t)]^{n_{i,1}} .$$

Using (6), our formula is now proved. Notice that the factor $\zeta_{f_{|\{l=0\}}}(t)$ corresponds to the disc A_0 defined in 1.8. $\square$

It is not hard to figure out how the process started in the proof above may continue. We apply Theorem 3.4 with h_f replaced by h'_δ and get a formula for the zeta-function $\zeta_{h'_\delta}(t)$, for any $\delta \in \mathscr{L}^{(1)}$. In a finite number of steps, going through the carrousel discs of order $1, 2, \ldots, m$, where $m := \max\{g_i \mid i \in \{1, \ldots, r\}\}$, we get a formula for $\zeta_f(t)$. To write it down, we need just the following notations.

Definition. Let $\delta(i)^{(k)}$ denote the carrousel disc of order k centred at a fixed (arbitrarily chosen) point of the set $\hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$. (This later set contains exactly $n_{i,1} \cdots n_{i,k}$ points). Denote $\mathscr{C}(\Delta') := \{\delta(i)^{(k)} \mid i \in \{1, \ldots, r\}, k \in \{1, \ldots, m\}\}$.

For any $\delta \in \mathscr{C}(\Delta')$, denote by $c(\delta)$ its centre and by $a(\delta)$ an arbitrarily chosen point on $\partial\bar{\delta}$.

Let $\delta \in \mathscr{C}(\Delta')$, where $\delta = \delta(i)^{(k)}$, for some indices i and k as above. Then define $n(\delta) := n_{i,1} \cdots n_{i,k}$.

Thus we get the following general zeta-function formula:

3.5. THEOREM. $\zeta_f(t) = \zeta_{f_{|\{l=0\}}}(t) \cdot \prod_{\delta \in \mathscr{C}(\Delta')} \zeta_{h'_{c(\delta)}}(t^{n(\delta)}) \cdot \zeta^{-1}_{h'_{a(\delta)}}(t^{n(\delta)})$. $\square$

By using a decreasing induction, $\zeta_f(t)$ will become finally a product of integer powers of cyclotomic polynomials.

3.6. *Remarks.* (a) The points $a(\delta), \delta \in \mathscr{C}(\Delta')$ may also be defined as follows (the precise details are left to the reader):

Let $\delta = \delta(i)^{(k)}$ and let $\hat{C}_i^{(k)}$ be (formally) defined by the equation (see (3)): $u_i = a_{k_i}\lambda^{m_{i,1}/n_{i,1}} + \cdots + \sum_{l=1}^{l_k} b_{k,l}\lambda^{(m_k+l)/n_{i,1}\cdots n_{i,k}}$. Then define a curve $G_{i,k}$, by slightly perturbing in this equation just the last coefficient b_{k,l_k}, such that $G_{i,k} \neq \hat{C}_j^{(k)}$, $\forall j \in \{1, \ldots, r\}$. For $k = g_i$, we cut the Puiseux expansion at a sufficiently high power of λ and modify the last coefficient. It follows that $a(\delta(i)^{(k)})$ may be identified to the point in $G_{i,k} \cap (D_\alpha \times \{\eta\})$ which is in the closest neighbourhood of $c(\delta(i)^{(k)})$.

(b) Let $\delta := \delta(i)^{(k)}$. Then $c(\delta)$ is a regular value for the map l_α if and only if, for any $j \in \{1, \ldots, r\}$ such that $\hat{C}_j^{(k)} = \hat{C}_i^{(k)}$, we have $g_j > k$. It is possible that $a(\delta)$ cannot be chosen arbitrarily close to $c(\delta)$, see also Remark 1.6.

(c) The carrousel monodromies $h'_{c(\delta)}$, $h'_{a(\delta)}$ may be defined as monodromies of functions. This remark was used by Lê in his proof of the Monodromy Theorem [Lê-1], see also [Lo]. For instance, if $\delta = \delta(i)^{(k)}$ and $(u_i^{(k)}(t), \lambda(t))$ is the parametrization of $\hat{C}_i^{(k)}$ defined in 1.5, then the pull-back diagram:

$$\begin{array}{ccc} (\mathbf{X}_i^{(k)}, 0) & \to & (\mathbf{X}, 0) \\ f_i^{(k)} \downarrow & & \downarrow \Phi \\ (\mathbf{C}, 0) & \xrightarrow{(u_i^{(k)}, \lambda)} & (\mathbf{C}^2, 0) \end{array} \tag{7}$$

defines a space $(\mathbf{X}_i^{(k)}, 0)$ and a function $f_i^{(k)}$ on it. Then $h'_{c(\delta)}$ is the monodromy of $f_i^{(k)}$.

3.7. We illustrate the formula on the following particular case: *any component* Δ_i *has just one Puiseux pair,* i.e. $g_i = 1$, $\forall i \in \{1, \ldots, r\}$. We assume, for the sake of simplicity, that the sets of components of $\Gamma(l, f)$ and $\Delta(l, f)$ are in one-to-one correspondence.

In this case, we have $\hat{C}_i^{(1)} = \Delta_i$ and a carrousel disc $\delta(i)^{(1)}$ is an arbitrarily small disc centred at $c(\delta(i)^{(1)}) \in \Delta_i \cap (D_\alpha \times \{\eta\})$, which is pointwise fixed by the $n_{i,1}$th iterate of the big carrousel. It follows that the point $a(\delta(i)^{(1)})$ can be chosen arbitrarily close to $c(\delta(i)^{(1)})$. The centres $c(\delta)$, $\delta \in \mathscr{C}(\Delta')$ are critical values for the map l_α. Let $c(i)$ denote a fixed, arbitrarily chosen point of the set $\Delta_i \cap (D_\alpha \times \{\eta\})$. Then $\mathscr{C}(\Delta')$ can be identified to the set $\{c(i) \mid i \in \{1, \ldots, r\}\}$. With these notations, the zeta-function formula becomes

$$\zeta_f(t) = \zeta_{f|\{l=0\}}(t) \cdot \prod_{i \in \{1, \ldots, r\}} \zeta_{h^{\text{rel}}_{c(i)}}(t^{n_{i,1}}), \tag{8}$$

where $h^{\text{rel}}_{c(i)}: H^\bullet(l_\alpha^{-1}(c(\delta)), l_\alpha^{-1}(a(\delta))) \circlearrowleft$ is the relative monodromy and its zeta-function is $\zeta_{h^{\text{rel}}_{c(i)}}(t) = \zeta_{h'_{c(\delta)}}(t)\zeta^{-1}_{h'_{a(\delta)}}(t)$. One also gets $\Lambda(f) = \Lambda(f_{|\{l=0\}}) + \Sigma_{i\in\{1,\dots,r\},\, n_{i,1}=1}\Lambda(h^{\text{rel}}_{c(i)})$.

By standard arguments, $H^\bullet(l_\alpha^{-1}(c(\delta)), l_\alpha^{-1}(a(\delta)))$ is isomorphic to the direct sum of reduced cohomologies $\oplus_{\upsilon\in l_\alpha^{-1}(c(\delta))\cap\Gamma}\tilde{H}^{\bullet-1}(F'_\upsilon)$, where $F'_\upsilon := B_{\upsilon,\varepsilon}\cap l_\alpha^{-1}(a(\delta))$ is the local Milnor fibre and $B_{\upsilon,\varepsilon}$ is a Milnor ball of the isolated singularity at υ. Let $d_i := \# l_\alpha^{-1}(c(i))\cap\Gamma$.

A point $\upsilon\in l_\alpha^{-1}(c(i))\cap\Gamma$ goes, after $n_{i,1}$ complete turns of the carrousel, to $\upsilon'\in l_\alpha^{-1}(c(i))\cap\Gamma$ and $\upsilon'\neq\upsilon$ if $n_{i,1}\geqslant 2$. After exactly $n_{i,1}d_i$ turns, the point υ is fixed.

It becomes clear how the relative monodromy acts on the above direct sum; by similar arguments as those in [Si, p. 192], one shows that the matrix of $h^{\text{rel}}_{c(i)}$ may be assumed to have the following block decomposition

$$\begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & \mathbf{V}_i\mathbf{T}_i^{n_{i,1}d_i} \\ \mathbf{I} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \mathbf{I} & \cdot & & & \cdot & 0 \\ \cdot & & \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & & \cdot & \cdot \\ \cdot & & & & \cdot & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & \mathbf{I} & 0 \end{bmatrix},$$

where, at some fixed $\upsilon(i)\in l_\alpha^{-1}(c(i))$, $\mathbf{I}$ is the identity matrix on $\tilde{H}^\bullet(F'_{\upsilon(i)})$, $\mathbf{T}_i$ is the *horizontal monodromy* of the transversal singularity and $\mathbf{V}_i$ is the *vertical monodromy* of the local system on $\Gamma_i\setminus\{0\}$, with fibre $\tilde{H}^\bullet(F'_{\upsilon(i)})$. Then $\zeta_{h^{\text{rel}}_{c(i)}}(t) = \Pi_{j\geqslant 0}\det[\mathbf{I} - t^{d_i}\mathbf{V}_i\mathbf{T}_i^{n_{i,1}d_i};\ \tilde{H}^j(F'_{\upsilon(i)})]^{(-1)^j}$. Finally, our formula looks as follows:

$$\zeta_f(t) = \zeta_{f|\{l=0\}}(t)\cdot\prod_{i\in\{1,\dots,r\}}\prod_{j\geqslant 0}\det[\mathbf{I} - t^{n_{i,1}d_i}\mathbf{V}_i\mathbf{T}_i^{n_{i,1}d_i};\ \tilde{H}^j(F'_{\upsilon(i)})]^{(-1)^j}. \tag{9}$$

3.8. This latter one may be easily specialized to the Siersma's formula [*loc. cit.*]. Let A_m be the most exterior annulus and assume that the components of Δ which cut A_m are $\Delta_1,\dots,\Delta_s$ and they have just one Puiseux pair. Denote $D_{m-1} := D_\alpha\times\{\eta\}\setminus A_m$. By our approach we get $\zeta_f(t) = \zeta_{h_{D_{m-1}}}(t)\cdot\Pi_{i\in\{1,\dots,s\}}\zeta_{h^{\text{rel}}_{c(i)}}(t^{n_{i,1}})$.

Let then g be a function with 1-dimensional singular locus $\Sigma = \cup_{i\in\{1,\dots,s\}}\Sigma_i$ and let $f := g + l^K$, for some $l\in\Omega_g$, with $K\in\mathbf{N}$ large enough. Then f is an isolated singularity and, as shown in [Si], one may identify the monodromy of the Milnor fibre F_g to $h_{D_{m-1}}$. The degree of the covering $\Sigma_i\setminus\{0\}\to\Delta_i\setminus\{0\}$ is d_i. Then one gets [Si, p. 183]:

$$\zeta_f(t) = \zeta_g(t)\cdot\prod_{i\in\{1,\dots,s\}}\det[\mathbf{I} - t^{Kd_i}V_i\cdot T_i^{Kd_i}]^{(-1)^{\dim(\mathbf{X},0)}}. \tag{10}$$

3.9. *Example.* Let $\mathbf{X} := \{x^3 + y^4 + z^3 = 0\} \subset \mathbf{C}^3$ and let $f \in \mathbf{m}_{\mathbf{X},0}$ be the function induced by $\tilde{f} \in \mathbf{m}_{\mathbf{C}^3,0}$, $\tilde{f} = x$. Consider the linear function l induced by $\tilde{l} = y$. Then $l \in \Omega_f$. We get that $\Delta(l, f)$ is irreducible and has the Puiseux parametrization: $l = \alpha v^3$, $\lambda = v^4$, where α is a nonzero constant, easy to determine.

Let $c \in \Delta(l, f) \cap (D_\alpha \times \{\eta\})$ and let $a \notin \Delta(l, f) \cap (D_\alpha \times \{\eta\})$ be a neighbour point of c.

The monodromy h'_a can be identified to the monodromy of the function $f_a : (\mathbf{X}_a, 0) \to (\mathbf{C}, 0)$ induced by $\tilde{f}_a = v$, where $\mathbf{X}_a := \{x = v^4, y = v^3, z = \sqrt[3]{2}\gamma v^4\}$ and γ is a 3-root of -1. We get $\zeta_{h'_a}(t) = (1-t)^{-3}$, hence $\zeta_{h_c^{\mathrm{rel}}} = (1-t)^2$.

By using (8), the final result is $\zeta_f(t) = (1-t)^{-3}(1-t^4)^2$. We also get $\Lambda(f) = 3$.

Notice that there is another way of computing the zeta function in this example, by using the usual $\mathbf{C}^*$-action on $\mathbf{X}$, which fixes the zero set $\{\tilde{f} = 0\}$. It follows that the monodromy h_f of f is equal to the 3rd power of the monodromy h_g of the function $g : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$, $g = y^4 + z^3$ and $\zeta_{h_g^3}(t)$ can be computed from the eigenvalues of h_g. If we change the above function $\tilde{f}$ into $\tilde{f}_1 := x + y$, then the set $\{\tilde{f}_1 = 0\}$ is no more invariant under the above-mentioned $\mathbf{C}^*$-action. The computations for the zeta-function of h_{f_1} are slightly more complicated, since we get two Puiseux pairs, with $n_{1,1} = 1$, $n_{1,2} = 3$. This time, the result is $\zeta_{f_1}(t) = (1-t)^{-1}(1-t^3)^{-1}(1-t^9)$.

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