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JACOBI FORMS AND SIEGEL MODULAR FORMS: RECENT RESULTS AND PROBLEMS

by Winfried KOHNEN

INTRODUCTION

In the present paper we would like to describe some recent developments how Jacobi forms can be used to study Siegel modular forms of genus 2 and what problems arise in this way. After a preliminary section on Siegel modular forms and Jacobi forms (§ 1) which mainly serves to fix some notation, we shall discuss the so called Maass space in § 2. We shall then study relations between Jacobi forms and spinor zeta functions of Hecke eigenforms of genus 2 (§ 3) and finally in § 4 will indicate how Jacobi forms can be used to give estimates for Fourier coefficients of Siegel cusp forms.

Sections 2-4 are divided into two parts: part one describes known results while part two gives some open problems.

We do not go here into any more intrinsic properties of Jacobi forms (as e.g. the trace formula or relations to modular forms of integral weight) nor discuss any representation-theoretic aspects of the theory. For good surveys, we refer to [33, 36] for the first and to [3] for the second topic.

§ 1. PRELIMINARIES ON SIEGEL MODULAR FORMS AND JACOBI FORMS

1.1. SIEGEL MODULAR FORMS OF GENUS 2

We write $\mathcal{H}_2$ for the Siegel upper half-space of genus 2. The natural action of $\mathrm{Sp}_2(\mathbf{R})$ on $\mathcal{H}_2$ is denoted by

$$(M, Z) \mapsto M<Z> := (AZ+B)(CZ+D)^{-1}$$

$$\left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbf{R}), \quad Z \in \mathcal{H}_2\right).$$

We put $\Gamma_2 := \mathrm{Sp}_2(\mathbf{Z})$ and for $k \in \mathbf{Z}$ denote by $M_k(\Gamma_2)$ the space of Siegel modular forms of weight k on Γ_2, i.e. the space of holomorphic functions $F: \mathcal{H}_2 \to \mathbf{C}$ satisfying $F(M<Z>) = \det(CZ+D)^k F(Z)$ for all $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$. Such a function has a Fourier expansion

$$F(Z) = \sum_{T = T' \geqslant 0} a(T) e^{2\pi \, \mathrm{itr}\,(TZ)}$$

where T runs over all positive semi-definite half-integral (2, 2)-matrices. We write $S_k(\Gamma_2)$ for the subspace of cusp forms (require $a(T) = 0$ for $T \not> 0$).

For $F, G \in S_k(\Gamma_2)$ we denote by

$$<F, G> = \int_{\Gamma_2 \backslash \mathcal{H}_2} F(Z) \overline{G(Z)} (\det Y)^{k-3} dX dY \qquad (X = \mathrm{Re}(Z), \;\; Y = \mathrm{Im}(Z))$$

the Petersson scalar product of F and G.

For basic facts on Siegel modular forms we refer to [12, 17].

1.2. Jacobi forms

We write $\mathcal{H}$ for the complex upper half-plane. We let $H(\mathbf{R})$ be the Heisenberg group, i.e. the set of triples $((\lambda, \mu), \kappa) \in \mathbf{R}^2 \times \mathbf{R}$ with group law $((\lambda, \mu), \kappa)((\lambda', \mu'), \kappa') = ((\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\mu' - \lambda'\mu)$, and denote by $G^J := SL_2(\mathbf{R}) \times H(\mathbf{R})$ the Jacobi group where $SL_2(\mathbf{R})$ operates on $H(\mathbf{R})$ from the right by $((\lambda, \mu), \kappa)M = ((\lambda, \mu)M, \kappa)$. The group G^J acts on $\mathcal{H} \times \mathbf{C}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left((\lambda, \mu), \kappa \right) \right) \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

We set $\Gamma_1 := SL_2(\mathbf{Z})$, $\Gamma_1^J := \Gamma_1 \times H(\mathbf{Z})$ and for $k \in \mathbf{Z}$ and $m \in \mathbf{N}_0$ denote by $J_{k,m}$ the space of Jacobi forms of weight k and index m on Γ_1^J, i.e. the space of holomorphic functions $\phi: \mathcal{H} \times \mathbf{C} \to \mathbf{C}$ satisfying the transformation formula

$$\phi(\gamma \circ (\tau, z)) = (c\tau + d)^k \exp\left(2\pi i m \left(\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} - \lambda^2\tau - 2\lambda z \right) \right) \phi(\tau, z)$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left((\lambda, \mu), \kappa \right) \right) \in \Gamma_1^J$ and having a Fourier expansion

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z}, r^2 \leqslant 4mn} c(n, r) q^n \zeta^r$$

where $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi i z}$. We write $J_{k,m}^{\mathrm{cusp}}$ for the subspace of cusp forms (require $c(n, r) = 0$ for $r^2 = 4mn$). Note that the coefficients $c(n, r)$ depend only on the discriminant $D := r^2 - 4mn$ and the residue class $r \pmod{2m}$.

The Petersson scalar product on $J^{\text{cusp}}_{k,m}$ is normalized by

$$<\phi, \psi> = \int_{\Gamma_1^J \backslash \mathscr{H} \times \mathbf{C}} \phi(\tau, z)\overline{\psi(\tau, z)} \exp(-4\pi m y^2/v) v^{k-3} du dv dx dy$$

$$(\tau = u + iv, \ z = x + iy) .$$

For basic facts about Jacobi forms we refer to [9].

§2. The Maass space

2.1. Results

Let F be a Siegel modular form of integral weight k on Γ_2 and write the Fourier expansion of F in the form

$$(1) \qquad F(Z) = \sum_{m \geqslant 0} \phi_m(\tau, z) e^{2\pi i m \tau'} \qquad \left(Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathscr{H}_2\right) .$$

Using the injection

$$(2) \qquad \Gamma_1^J \to \Gamma_2 , \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa)\right) \mapsto \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the transformation formula of F it is easy to see that the functions ϕ_m are in $J_{k,m}$. The expansion (1) is referred to as the Fourier-Jacobi expansion of F.

Thus for any $m \in \mathbf{N}_0$ we obtain a linear map

$$(3) \qquad \rho_m : M_k(\Gamma_2) \to J_{k,m} , \ F \mapsto \phi_m .$$

Note that ρ_0 is equal to the Siegel Φ-operator.

We shall be interested in the case $m = 1$. For k odd, ρ_1 is the zero map; in fact, any Jacobi form of odd weight and index one must vanish identically as is easily seen.

For k even, ρ_1 was studied in detail by Maass [28, 29] who showed the existence of a natural map $V: J_{k,1} \to M_k(\Gamma_2)$ such that the composite $\rho_1 \circ V$ is the identity. More precisely, let $\phi \in J_{k,1}$ with Fourier coefficients $c(n, r)$ $(n, r \in \mathbf{Z}; r^2 \leqslant 4n)$ and for $m \in \mathbf{N}_0$ define