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Artikel: JACOBI FORMS AND SIEGEL MODULAR FORMS: RECENT RESULTS AND PROBLEMS

Kapitel: §4. ESTIMATES FOR FOURIER COEFFICIENTS OF SIEGEL CUSP FORMS

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§4. Estimates for Fourier coefficients of Siegel cusp forms

4.1. Results

Very recently, it has turned out that Jacobi forms can be used in a rather simple way to prove growth estimates for Fourier coefficients of Siegel cusp forms of genus 2. The bounds one obtains in this way, in fact, are somewhat better than those obtained previously by different methods.

Let F be a Siegel cusp form of integral weight k on Γ_2 and let $a(T)$ be its Fourier coefficients. The classical Hecke argument immediately gives

$$a(T) \ll_F (\det T)^{k/2} . \tag{7}$$

If one applies a classical theorem of Landau [25, 32] to the Rankin-Dirichlet series

$$\sum_{\{T>0\}/GL_2(\mathbf{Z})} |a(T)|^2 (\det T)^{-s}$$

where the summation extends over a complete set of representatives for the usual left-action of $GL_2(\mathbf{Z})$ on the set of positive definite symmetric half-integral (2, 2)-matrices T, one can sharpen (7) and show that

$$a(T) \ll_{\varepsilon,F} (\det T)^{k/2-3/32+\varepsilon} \qquad (\varepsilon > 0) .$$

(Recall that Landau's theorem roughly speaking asserts that if a Dirichlet series has a meromorphic continuation to $\mathbf{C}$ and satisfies an appropriate functional equation, then one can deduce a "good" upper bound for the growth of its coefficients.) For details we refer to [5] and also [11] where the argument is slightly different; note that the authors prove an estimate for arbitrary genus n.

Let us mention the following

Theorem 1 (Kitaoka [16]). *Suppose that* k *is even. Then*

$$a(T) \ll_{\varepsilon,F} (\det T)^{k/2-1/4+\varepsilon} \qquad (\varepsilon > 0) .$$

The proof of Theorem 1 uses Poincaré series of exponential type on Γ_2 and estimates for generalized matrix-argument Kloosterman sums and can be viewed as a generalization to genus 2 of a well-known method how to obtain "good" bounds for the Fourier coefficients of elliptic cusp forms.

Let us explain now briefly how Jacobi forms can be brought into play (for full details cf. [20, 21]). Let $\phi \in J_{k,m}^{\text{cusp}}$ with Fourier coefficients $c(n, r)$. Then for $k > 2$ one shows that

(8) $$c(n, r) \ll_{\varepsilon, k} (m + |D|^{1/2+\varepsilon})^{1/2} \frac{|D|^{k/2-3/4}}{m^{(k-1)/2}} \| \phi \| \qquad (\varepsilon > 0)$$

where $D := r^2 - 4mn$ and the bound in $\ll$ only depends on ε and k.

For the proof one carries over the method of Poincaré series and Kloosterman sums from the one-variable situation already mentioned above to the case of the Jacobi group. Note that Poincaré series on Γ_1^J were studied in [14, II, §2]. The Kloosterman sums that occur in their Fourier coefficients can be related to Salié sums and therefore can easily be estimated (a similar phenomenon happens in the case of modular forms of half-integral weight, cf. [15]). The proof of (8) for D a fundamental discriminant (i.e. the discriminant of a quadratic field) is given in [20, §1] and for arbitrary D is given in [21, §1].

On the other hand, if one applies Landau's theorem to the Dirichlet series $D_{F,F}(s)$ discussed in §3, one finds that

(9) $$\| \phi_m \| \ll_{\varepsilon, F} m^{k/2-2/9+\varepsilon} \qquad (\varepsilon > 0) .$$

The estimates (8) and (9) now imply the following

THEOREM 2 [20, 21]. *One has*

(10) $$a(T) \ll_{\varepsilon, F} (\det T)^{k/2-13/36+\varepsilon} \qquad (\varepsilon > 0) .$$

In fact, both sides of (10) are invariant under $T \mapsto U'TU (U \in GL_2(\mathbf{Z}))$, hence if in (10) we write $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, then we may assume that $m = \min T$, where $\min T$ denotes the least positive integer represented by T. If we use (8) and (9) together with the fact that $\min T \ll (\det T)^{1/2}$ which is well-known from reduction theory, we obtain (10).

4.2. PROBLEMS

i) In [15], Iwaniec using some sophisticated arguments for certain sums of Salié sums showed that the Fourier coefficients $a(n)$ $(n \in \mathbf{N})$ of a cusp form f of weight $k - 1/2$ for $k > 0$ and n squarefree satisfy

$$a(n) \ll_K \sigma_0(n) (\log 2n)^2 n^{k/2-15/28} \| f \| ,$$

where $\sigma_0(n)$ is the number of positive divisors of n and $\| f \|$ is the appropriately normalized Petersson norm of f. We wonder if it is possible to prove an analogous estimate for the Fourier coefficients $c(n, r)$ $(D = r^2 - 4mn$ a fundamental discriminant) of a function $\phi \in J_{k,m}^{\text{cusp}}$ for $k > 2$ which also is analogous to (8) in the sense that an appropriate power of m appears in the denominator on the right-hand side. This then eventually would lead to some improvement of (10) in the case where $-4 \det T$ is a fundamental discriminant.

ii) For arbitrary genus n, the best estimate for Fourier coefficients so far known is due to Böcherer and Raghavan [5] and independently Fomenko [11]. Using Rankin's method they showed that the Fourier coefficients $a(T)$ (T a positive definite symmetric half-integral (n, n)-matrix) of a cusp form F of integral weight k on Γ_n satisfy

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2 - \delta_n + \varepsilon} \qquad (\varepsilon > 0)$$

where $\delta_n := 2n + 2 + 4\left[\frac{n}{2}\right] + \frac{2}{n+1}$ and $[x]$ = integral part of x (the case $n = 2$ was discussed above).

It is natural to try to apply the method described in 4.1 also for higher genus n. For some results in this direction we refer to [7].

iii) Let F be a non-zero Hecke eigenform in $S_k(\Gamma_2)$ with eigenvalues λ_n and Fourier coefficients $a(T)$.

If k is even and F is in the Maass space $S_k^*(\Gamma_2)$, then

$$\lambda_n \ll_\varepsilon n^{k-1+\varepsilon} \qquad (\varepsilon > 0) \tag{11}$$

and this estimate is best possible as follows from Theorem 2 in §2.

On the other hand, if k is odd or if k is even and $F \in S_k^*(\Gamma_2)^\perp$, then one expects that the generalized Ramanujan-Petersson conjecture holds which predicts that

$$\lambda_n \ll_\varepsilon n^{k-3/2+\varepsilon} \qquad (\varepsilon > 0)\,. \tag{12}$$

To the author's knowledge, the best estimate proved so far for the numbers λ_n is due to Duke, Howe and Li [8] who showed using representation-theoretic methods that

$$\lambda_n \ll_\varepsilon n^{k-1+\varepsilon} \qquad (\varepsilon > 0) \tag{13}$$

provided that n is squarefree (in fact, under the assumption n squarefree the authors proved that $\lambda_n \leqslant \sigma_0(n)^2 n^{k-1}$; it is suggestive that their method, in fact, gives (13) for all n).

In [1, Chap. 2] Andrianov proved that if D is a negative fundamental discriminant and $T_1, \ldots, T_h$ $(h = h(D))$ denotes a set of Γ_1-representatives of positive definite symmetric half-integral (2, 2)-matrices with discriminant D, then

$$\zeta_{\mathbf{Q}(\sqrt{D})}(s-k+2) \sum_{\nu=1}^{h} \left(\sum_{n \geqslant 1} a(nT_\nu) n^{-s} \right) = \left(\sum_{\nu=1}^{h} a(T_\nu) \right) Z_F(s)\,. \tag{14}$$

Thus – roughly speaking – for fixed T the eigenvalues λ_n are "proportional" to the coefficients $a(nT)$.

Suppose that F is in $S_k^*(\Gamma_2)$. Using Theorem 1 in §2 and the estimate (8) with $m = 1$ one finds that

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2 - 1/2 + \varepsilon} \quad (\varepsilon > 0)\,, \tag{15}$$

and (11) together with (14) implies that (15), in fact, is best possible.

On the other hand, taking into account (12) and the fact that the Hecke eigenforms form a basis, one may be led to the following

CONJECTURE 1 [11]. *Let F be a cusp form of integral weight k on Γ_2 and suppose that either k is odd or that k is even and F is in the orthogonal complement of the Maass space. Let $a(T)$ (T a positive definite symmetric half-integral* (2, 2)*-matrix) be the Fourier coefficients of F. Then*

$$a(T) \ll_{\varepsilon, F} (\det T)^{k/2 - 3/4 + \varepsilon} \quad (\varepsilon > 0)\,.$$

Concerning norms of Fourier-Jacobi coefficients, one optimistically may hope for the truth of the following

CONJECTURE 2. *Let F be a cusp form of integral weight k on Γ_2 and denote by ϕ_m ($m \in \mathbf{N}$) its Fourier-Jacobi coefficients. Then*

$$\| \phi_m \| \ll_{\varepsilon, F} m^{k/2 - 1/2 + \varepsilon} \quad (\varepsilon > 0)\,. \tag{16}$$

Note that – in view of Theorem 1 of §3 – (17) would be best possible.

NOTE ADDED IN PROOF

The estimates for the Fourier coefficients of cusp forms of arbitrary genus $n \geqslant 2$ obtained in [7] improve upon those obtained in [5, 11], cf. 4.2. ii).

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