4. The cylindrical trace

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

Then $\beta(f_1)$ is represented on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$

$$\text{by the matrix } \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}$$

using a symplectic basis of $\mathbf{C}^2$, and $\beta(u)$ is the obvious cyclic permutation on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$. But then $2 - \beta(f_1)$ is the transposition on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ exchanging the first two copies of $\mathbf{C}^2$. Thus the image of $\mathscr{A}(n, \delta)$ is the same as that of the group algebra of the symmetric group.

4. The cylindrical trace

There is a natural trace functional tr on $A(n, \delta)$ defined by $\mathrm{tr}(D) = \delta^{n(D)}$, $n(D)$ being the number of closed loops formed on the cylinder if the inside and outside boundaries of the annulus are identified. We will call this trace the cylindrical trace.

Note 4.1. This trace exists in fact on the whole Brauer algebra — it could be defined in terms of partitions as $\mathrm{tr}(D) = \delta^{n(D)}$ where $n(D)$ is the number of equivalence classes for the equivalence relation generated by D itself and the relation which identifies each point on the top with the corresponding point on the bottom.

Note 4.2. One has the relation $n(D_1 \circ D_2) = n(D_2 \circ D_1)$ so one might try to define a more general trace by replacing δ by an arbitrary complex number. But $n(\alpha, \beta) \neq n(\beta, \alpha)$ in general so one is forced to choose δ.

If δ is a value for which $A(n, \delta)$ is semisimple we know that $A(n, \delta)$ is a direct sum of matrix algebras, so our cylindrical trace is determined by its value on a minimal idempotent in each matrix algebra summand. We will calculate these "weights" of the trace. In order to do this we will need detailed information on the multiplicities of u in each irreducible representation of $A(n,\delta)$.

Definition 4.3. For $n \geqslant t > 0$ the group $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/t\mathbf{Z} (= \{(a, b) \mid a = 0, \ldots, n-1; b = 0, \ldots t-1\}$ acts by linear transformations on $\mathscr{A}(t, n; t)$ by $(a, b)(D) = u^a \circ D \circ u^b$. (The u's on the left and right in this formula are of course different if $n \neq t$.) Let $F_{n,t}(a, b)$ be the number of fixed points for (a, b). Let $F_n(a)$ be the number of fixed points for the action $D \mapsto D \circ u^a$ of $a \in \mathbf{Z}/n\mathbf{Z}$ on $\mathscr{A}(n, 0)$.

LEMMA 4.4. *The multiplicity of an n-th root of unity* η *as an eigenvalue of* u *in the representation* $\pi_{t,\omega}$ $(\omega^t = 1)$ *is* $\frac{1}{nt} \sum_{a=0}^{n-1} \sum_{b=0}^{t-1} \eta^{-a}\omega^{-b}F_{n,t}(a,b)$ *for* $t > 0$ *and* $\frac{1}{n}\sum_{a=0}^{n-1} \eta^a F_n(a)$ *for* $t = 0$.

Proof. From the definition of $\pi_{t,\omega}$ it is clear that the multiplicity is trace $\left(p_\omega\left(\frac{1}{n}\sum_{a=0}^{n-1} \eta^{-a}u^a\right)\right)$. $\square$

Definition 4.5. For each t-th root of unity ω let $\mathcal{M}(\omega, n)$ (or $\mathcal{M}_{t,n}(\omega, n)$ if it is necessary to specify that ω is indeed a t-th root of unity and not some other) be the cylindrical trace of a minimal projection in the simple summand of $A(n, \delta)$ corresponding to $\pi_{t,\omega}$. To determine $\mathcal{M}(\omega, n)$ we will use the following easy result.

LEMMA 4.6. *For each* $0 \leqslant r < n, r + n$ *even, there is an algebra isomorphism* $\varphi: A(r, \delta) \to e_r A(n, \delta) e_r$ *such that*

(1) $\mathrm{tr}(\varphi(x)) = \mathrm{tr}(x)$, $x \in A(r, \delta)$.

(2) If p *is a minimal projection in the summand of* $A(r, \delta)$ *indexed by* (t, ω), $t \leqslant r$, *then* $\varphi(p)$ *is a minimal projection in the summand of* $A(n, \delta)$ *indexed by* (t, ω).

Proof. Define φ on diagrams by $\varphi(D) = \delta^{\frac{r-n}{2}} D'$, D' differing from D by first inserting $n - r$ interior and exterior points to the right of $*$ and connecting them up in adjacent pairs, very close to the boundary so as to not interfere with the rest of the diagram. Then move $*$ one to the right to ensure that the identity of $A(r, \delta)$ is mapped onto the element we have called e_r. The process of constructing D' from D is illustrated in Figure 4.7.

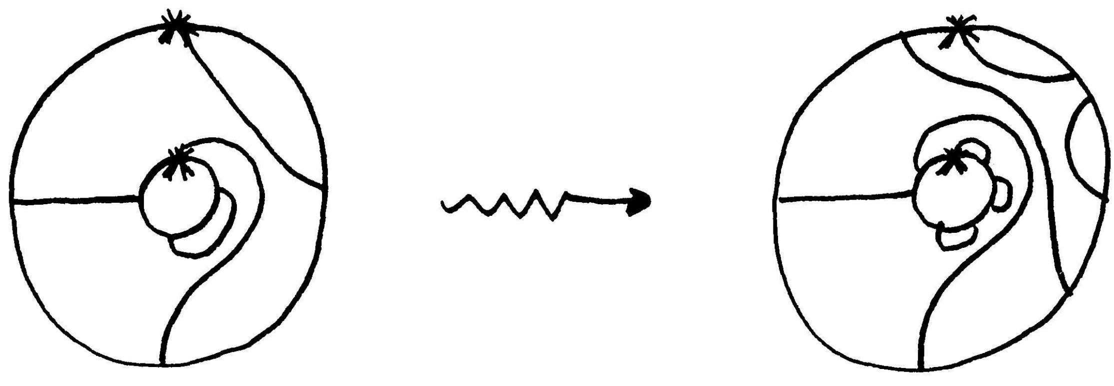

FIGURE 4.7

When closed on the cylinder D' will have exactly $\frac{n-r}{2}$ more closed loops than D so $\mathrm{tr}(\phi(x)) = \mathrm{tr}(x)$. The multiplicativity of ϕ also follows from the factor $\delta^{\frac{r-n}{2}}$ in its definition. Injectivity of ϕ is obvious and surjectivity follows by considering a diagram of $E_r \circ D \circ E_r$ for $D \in \mathcal{A}(n, n)$.

Finally, for $t \leqslant r$, $\phi(v_t) = v_t$ (with an obvious abuse of notation) and $\phi(e_t) = e_t$. The summand of $A(r, \delta)$ or $A(n, \delta)$ indexed by (t, ω) is characterized by $v_t = \omega e_t$ (when multiplied by a minimal central idempotent corresponding to the summand). □

We are now in a position to give a formula that determines $\mathcal{M}(\omega, n)$.

THEOREM 4.8. *For* $r < n$, $\mathcal{M}_{r,n}(\omega, n) = \mathcal{M}_{r,r}(\omega, r)$ *and, if* $r = n$,

$$\mathcal{M}(\eta, n) = \frac{1}{n} \sum_{j=1}^{n} \delta^{GCD(j,n)} \eta^{j} - \sum_{\substack{n > t > 0 \\ t + n \text{ even}}} \sum_{\omega, \omega^t = 1} \mathcal{M}(\omega, t) \left\{ \frac{1}{nt} \sum_{a=0}^{n-1} \sum_{b=0}^{t-1} \eta^{-a} \omega^{-b} F_{n,t}(a, b) \right\} - \frac{1}{n} \sum_{a=0}^{n-1} \eta^{-a} F_n(a) .$$

Proof. Since the ϕ of Lemma 4.6 is surjective, a minimal idempotent in $A(r, \delta)$ is minimal in $A(n, \delta)$ for $r < n$, so by 4.6 we are reduced to the case $r = n$. If we fix an n-th root of unity η, the trace we are trying to calculate is $\operatorname{tr}(P \frac{1}{n} \sum_{j=1}^{n} \eta^j u^j)$ where $(1 - P)$ is the central idempotent of $A(n, \delta)$ corresponding to all matrix summands indexed by (t, ω) with $t < n$. Since the trace of u^j itself is clearly $\delta^{GCD(j,n)}$ one has

$$\mathcal{M}(\eta, n) + \operatorname{tr}\left((1 - P) \frac{1}{N} \sum_{j=1}^{n} \eta^{-j} u^j\right) = \frac{1}{n} \sum_{j=1}^{n} \delta^{GCD(j,n)} \eta^{-j} .$$

Writing $(1 - P)A(n, \delta)(1 - P)$ as a sum of matrix algebras we get the result by 4.4. □

Thus we only need to determine $F(a, b)$ and $F_n(a)$.

THEOREM 4.9. *If* $a = 0, 1, \ldots, n - 1$, $b = 0, 1, \ldots, t - 1$ $(n \geqslant t,\ t \neq 0)$, *let* $x = GCD(a, n)$, $y = GCD(b, t)$, *then*

(a)
$$F_{n,t}(a, b) = \begin{cases} 0 & \text{if } \frac{a}{x} \neq \frac{b}{y} \text{ or } \frac{b}{x} \neq \frac{t}{y} \text{ or } x \neq y \bmod 2 \\ t \dbinom{x}{\frac{x-y}{2}} & \text{otherwise (and } a, b \neq 0) \\ 0 & \text{if } a \text{ or } b = 0, \text{ not both, or } n + t \text{ odd} \\ t \dbinom{n}{\frac{n-t}{2}} & \text{if } a = b = 0 , \end{cases}$$

$$\text{(b)}\quad F_n(a) = \begin{cases} \frac{2m+1}{m+1}\binom{2m}{m} \left(= \binom{2m+1}{m} \right) & \textit{if } \; n = 4m+2 \; \textit{ and } \; a = 2m+1 \\ \binom{x}{\frac{x}{2}} & \textit{if } \; x \; \textit{ is even} \\ \frac{1}{n/2+1}\binom{n}{n/2} & \textit{if } \; a = 0 \; \textit{ and } \; n \; \textit{ is even} \\ 0 & \textit{otherwise}\,. \end{cases}$$

Proof. Let us prove (b) first as the method is the same for (a) but (b) is simpler.

In the case $n = 4m + 2$, we first claim that for a fixed diagram some point on the boundary must be joined to the point diametrically opposite. This is easy by induction — it is trivial for $n = 2$, and if $n > 2$, just choose two boundary points connected to each other. Either they are diametrically opposite each other and we are done, or the disc is divided into three regions as in Figure 4.10.

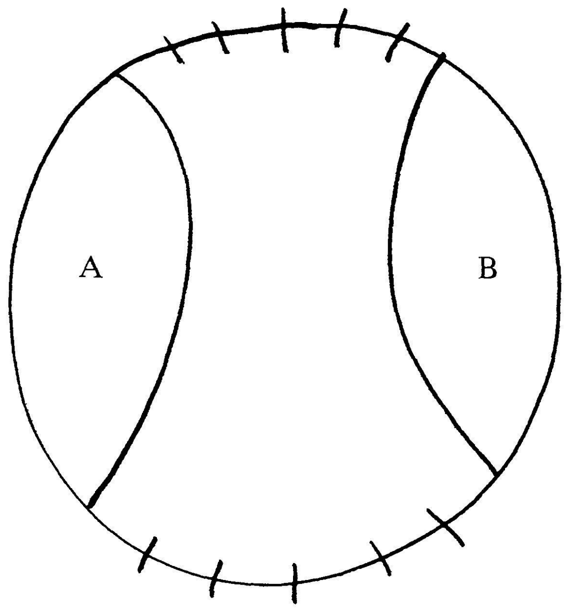

FIGURE 4.10

The boundary points inside A (hence B) are even in number so the number of marked boundary points in the diagram is congruent to 2 mod 4. But the original 180° rotation acts by a 180° rotation on these points so we are done by induction.

Once we know that some point is connected to a diametrically opposite point, the whole diagram, since it is fixed by the rotation of 180°, is determined by the configuration in one half. There are $\frac{1}{m+1}\binom{2m}{m}$ such configurations, and the diameter can be chosen in $2m + 1$ ways.

Now suppose $x = GCD(n, a)$ is even. Then the n boundary points may be divided up into n/x fundamental domains, each consisting of x consecutive points on the boundary. The x points in a fundamental domain can be divided into ones connected to points within the domain and ones connected to points in other fundamental domains. Moreover the constraint of planarity clearly implies that if a point is connected to a point in another fundamental domain, that other domain must be adjacent to it. Thus we may speak of clockwise and anticlockwise points and obviously, since the diagram is fixed, there are the same number of clockwise as anticlockwise points for each domain. We see that the whole diagram is completely determined by a single configuration as in Figure 4.11.

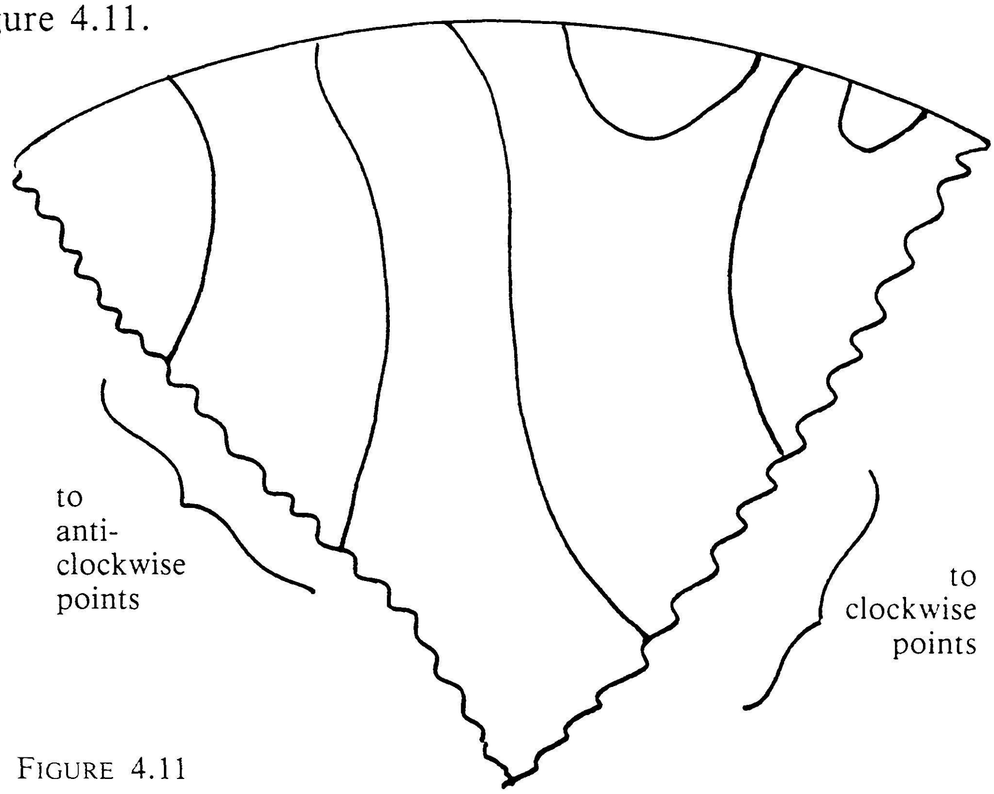

FIGURE 4.11

Also any such configuration determines a fixed point. Straightening out the wavy radii into a single straight line we see that these configurations are in bijection with $\cup_{i=0}^{x/2} \mathscr{P}(x, 2i; 2i)$ which has order $\binom{x}{\frac{x}{2}}$ by Lemma 1.12.

Finally for part (b), if x is odd, there would be an odd number of points in a fundamental domain, which is clearly impossible by the above argument.

Proof of (a). As in the proof of part (b), divide the n outside points into n/x "fundamental domains" for the rotation of a units on the outside circle. Each of the x points in a domain is then of one of four kinds: a "through-point" — attached to the inner circle; a clockwise point — attached to the adjacent domain in clockwise order; an anticlockwise point — similarly; or an internal point — attached to another point in the domain.

The whole system of connections can then be extended to all the outer points by rotating the fundamental domain by powers of the rotation of a units. The through-points can then be connected to the inside points in any of t ways which accounts for the factor of "t" in the formula. That any fixed diagram must look like this follows by arguing only on the outside points. The diagram will then be fixed by (a, b) if and only if the rotation through a points on the outside effects a rotation of b points when restricted to the through-points.

Now suppose there are r through-points per fundamental domain. Obviously $r \cdot \frac{n}{x} = t$, and the rotation of a effects a rotation of $\frac{ra}{x}$ on the through-points. Thus we must have $\frac{ra}{x} = b$, $r\frac{n}{x} = t$. Moreover the through-points in a fundamental domain must be connected to inner points in a fundamental domain for the rotation of b, so $y = r$. So the conditions $\frac{a}{x} = \frac{b}{y}$ and $\frac{n}{x} = \frac{t}{y}$ are necessary for a fixed point. The equality of x and y mod 2 follows from the fact that there have to be as many clockwise points as anticlockwise (as in part (b)) and the number of internal points is necessarily even.

Finally, if all the conditions are satisfied, any configuration as below can be extended in t ways to a fixed point for (a, b).

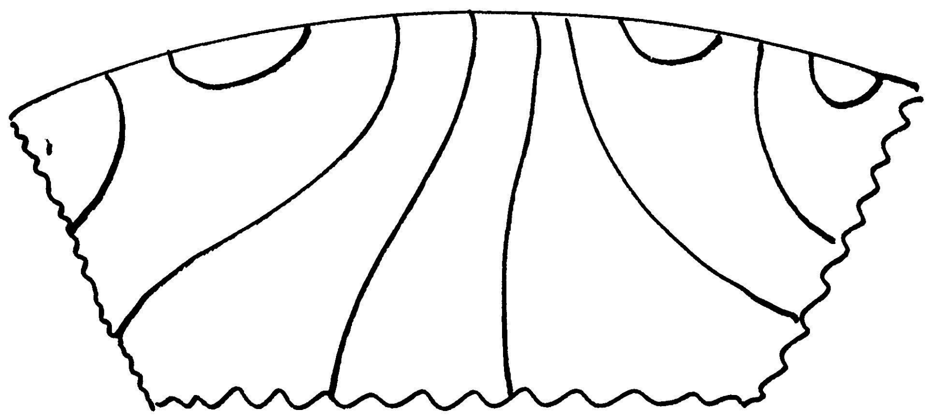

FIGURE 4.12: Two clockwise, two through and six internal points.

As in part (b), make the wavy line one straight line and we see there are $\binom{x}{\frac{x-y}{2}}$ such configurations by Lemma 1.12. $\square$

Given the apparently erratic nature of $F(a, b)$, the elegance of the final formula for $\mathcal{M}(\eta, n)$ seems to us quite remarkable. It will be most transparent if we use the Fourier transform. These are characters rather than multiplicities.

Definition 4.13. For $r = 0, 1, \ldots, n-1$, let $M(r, n) = \sum_\eta \eta^r \mathcal{M}(r, n)$, the sum being taken over all n-th roots of unity η.

The following result was first obtained on a computer by S. Eliahou.

THEOREM 4.14. *If* $T_n(x)$ *is the usual Tchebychev polynomial,* $T_n(\cos\theta) = \cos n\theta$, *then we have*

$$M(0,2) = \delta^2 - 1 = 2T_2\left(\frac{\delta}{2}\right) + 1$$

$$M(1,2) = \delta - 1 = 2T_1\left(\frac{\delta}{2}\right) - 1$$

and for $n > 2$,

$$M(r,n) = 2T_{GCD(n,r)}\left(\tfrac{\delta}{2}\right) .$$

Proof. Let us first obtain the recursive formula for $M(r,n)$ from Theorem 4.8:

$$\sum_{\eta:\eta^n=1} \mathcal{M}(\eta,n)\eta^r = \delta^{GCD(r,n)} - \left\{ \sum_{\substack{t>0 \\ t+n \text{ even}}}^{n-2} \sum_{\omega:\omega^t=1} \mathcal{M}(\omega,t)\frac{1}{t}\sum_{b=0}^{t-1} \omega^{-b}F_{n,t}(r,b)\right\}$$
$$- F_n(r)$$
$$= \delta^{GCD(r,n)} - \sum_{\substack{t>0 \\ t+n \text{ even}}}^{n-2} \frac{1}{t}\sum_{b=0}^{t-1} M(t-b,t)F_{n,t}(r,b) - F_n(r) ,$$

where the last term is only present if n is even.

We must show that the function defined in the statement of the theorem, call it $\mu(r,n)$, satisfies this recursion equation. Note first that if we set $P_n(x) = 2T_n\left(\frac{x}{2}\right)$ then (see [Lu]):

$$P_n(x) = x^n - \sum_{\substack{0\leqslant i<n \\ i+n \text{ even}}} \binom{n}{\frac{n-i}{2}} P_i(x) .$$

The case $r = 0$ is now rather easy: For $n = 1$, $M(0,1) = \delta - F_1(0) = \delta$ since there are no diagrams with one boundary point. Also $\mu(0,1) = P_1(\delta) = \delta$. For $n = 2$, $M(0,2) = \delta^2 - F_2(0) = \delta^2 - 1 = \mu(0,1)$. For $n > 2$ we have (first for n even).

$$\mu(0,n) = P_n(\delta) = x^n - \sum_{\substack{n>t>2 \\ t \text{ even}}} \binom{n}{\frac{n-t}{2}} \mu(0,t) + \binom{n}{\frac{n-2}{2}}(\delta^2-2) + \binom{n}{\frac{n}{2}}$$

$$= x^n - \sum_{\substack{n>t>2 \\ t \text{ even}}} \frac{1}{t} F_{n,t}(0,0)\mu(0,t) + \tfrac{1}{2}F_{n,2}(0,0)(\delta^2-1) + \binom{n}{\frac{n}{2}} - \binom{n}{\frac{n-1}{2}}$$

but since $F_{n,t}(0, b) = 0$ for $b \neq 0$, we get

$$\mu(0, n) = x^n - \sum_{\substack{n > t > 0 \\ t \text{ even}}} \frac{1}{t} \sum_{b=0}^{t-1} F_{n,t}(0, b)\mu(t-b, t) - F_n(0) .$$

The case where n is odd is even easier.

Now consider the case where r is arbitrary. Since $GCD(b, t) = GCD(t-b, t)$ we must show

$$\mu(r, n) = \delta^{GCD(r,n)} \sum_{\substack{t > 0 \\ t+n \text{ even}}}^{n-2} \frac{1}{t} \sum_{b=0}^{t-1} \mu(b, t) F_{n,t}(r, b) - F_n(r) .$$

Let us find all pairs (t, b), $0 < b \leqslant t-1$, $0 < t < n$, for which $F_{n,t}(r, b) \neq 0$. Let $g = GCD(r, n)$. Then from 4.6 we must have $t = \frac{\alpha n}{g}$, $b = \frac{\alpha r}{g}$, $\alpha + g$ even and $\alpha < g$, for $\alpha = GCD(r, n)$. On the other hand, if we are given an α with $\alpha + g$ even and $0 < \alpha < g$, then $GCD\left(\frac{\alpha n}{g}, \frac{\alpha r}{g}\right) = \alpha$ so if we put $t = \alpha n/g$, $b = \alpha r/g$, $0 < t < n$, $0 < b < t$ and $GCD(t, b) = \alpha$. Thus since $F_{n,t}(r, 0) = 0$ for $r \neq 0$, the equation to check becomes

$$\mu(r, n) = \delta^g - \sum_{\substack{\alpha > 0 \\ \alpha + g \text{ even}}}^{g-2} \mu\left(\frac{\alpha r}{g}, \frac{\alpha n}{g}\right) \binom{g}{\frac{g-\alpha}{\alpha}} - F_n(r) .$$

On the other hand,

$$P_g(\delta) = \delta^g - \sum_{\substack{\alpha > 0 \\ \alpha + g \text{ even}}}^{g-2} P_\alpha(\delta) \binom{g}{\frac{g-\alpha}{2}} - \binom{g}{\frac{g}{2}},$$

where the last term is present in the even case (for g) only.

Thus we are done if g is odd since then $F_n(r) = 0$ and there is no difference between these recursion relations. The sum in the expression for μ can only contain $\mu(1, 2)$ if $n = 2r$ so $g = r$ and $\alpha = 1$. In this case g is odd, so we are done in the case g even since then $F_n(r) = \binom{g}{\frac{g}{2}}$ *and* $\mu(a, b) = \mathscr{P}_{GCD(a,b)}$ by definition for all terms in the sum for μ. Finally there is the case $n = 2r$ $g(=r)$, odd. Then all the terms in the recursions are the same except the last two — for $\mu(r, n)$ we have

$$\mu(1, 2)\binom{g}{\frac{g-1}{2}} + \binom{g}{\frac{g-1}{2}} = (P_1 - 1)\binom{g}{\frac{g-1}{2}} + \binom{g}{\frac{g-1}{2}} = P_1\binom{g}{\frac{g-1}{2}}$$

which is the same as the last term in the formula for $P_g(\delta)$. □

COROLLARY 4.15. *The traces of minimal idempotents in the matrix algebra summand corresponding to* (ω, t), $\mathcal{M}(\omega, t)$, *are given by:*

For $t = 1$, $\mathcal{M}(1, 1) = 1$

For $t = 2$, $\mathcal{M}(1, 2) = \dfrac{\delta^2 + \delta - 2}{2}$, $\mathcal{M}(-1, 2) = \dfrac{\delta^2 - \delta}{2}$

For $t > 2$,
$$\mathcal{M}(\omega, t) = \frac{1}{t} \sum_{r=0}^{t-1} 2T_{GCD(r,n)}\left(\frac{\delta}{2}\right) \omega^r$$
$$= \frac{2}{t} \sum_{d \mid t} \left(\sum_{\substack{k: GCD(n,k) = d \\ k \leqslant n}} \omega^k \right) T_k\left(\frac{\delta}{2}\right).$$

Proof. Just invert the Fourier transform. □

COROLLARY 4.16. *The multiplicity of the representation* $\pi_{t,\omega}$ *of* $A(n, k)$ *in the Brauer representation* β *(§3) is* $\mathcal{M}(\omega, t)(k)$, *for* $k \geqslant 3$. *(So* $\mathcal{M}(\omega, k) > 0$ *for* $k \geqslant 3$.*)*

Proof. For $k \geqslant 3$ the algebra is semisimple and the trace induced by the usual trace of $\mathrm{End}(\otimes^n V)$ is the cylindrical trace, with parameter $\delta = k$. □

If we look at the oriented subalgebra $\overrightarrow{A(n, \delta)}$ (with n even), the irreducible representations are parametrised by even t's and the first $t/2$ t-th roots of unity ω. Obviously $\mathcal{M}(\omega, t) = \mathcal{M}(\bar{\omega}, t)$ since $GCD(r, n) = GCD(n-r, n)$. Let $\overrightarrow{\mathcal{M}(\omega, t)}$ denote the cylindrical trace of a minimal idempotent in the summand corresponding to $\pi(t, \omega)$.

COROLLARY 4.17. $\vec{\mathcal{M}}(\omega, t) = \frac{4}{t} \sum_{r=0}^{t-1} T_{GCD(2,n)}\left(\frac{\delta}{2}\right) \omega^r$, *for* $n > 2$.

Proof. *On restriction to* $\vec{A}(n, \delta)$ the representations of A parametrised by ω and $-\omega$ become equivalent.

COROLLARY 4.18. *The Brauer representation* β *is not faithful for* $k = 2$ *and* $n \geqslant 3$.

Proof. $T_n(1) = 1$ so for $\omega \neq 1$, $\mathcal{M}(\omega, t) = 0$, and this is sufficient to imply that the matrix algebra corresponding to ω is in the kernel of β.