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$V_1$ -invariant subsets of  $X$  and  $v_2(t_k)Y_k \subseteq X$  for all  $k$ . Also it is easy to see that  $\{t_k | k \geq 1\}$  contains either all positive rational or all negative rational numbers. Now let  $Y' = \bigcap_{k=1}^{\infty} Y_k$ . Since  $\{Y_k\}$  is a decreasing sequence of compact subsets,  $Y'$  is nonempty. Now if  $\{t_k | k \geq 1\}$  contains all positive rational numbers then  $v_2(r)Y' \subseteq X$  for all positive rational numbers  $r$  and hence by continuity  $V_2^+ Y' \subseteq X$  and, similarly, in the alternative case  $V_2^- Y' \subseteq X$ . This completes the proof of the theorem.

#### APPENDIX: RECURRENT POINTS

For a compact metric space  $X$  we denote by  $C(X)$  the space of all continuous real-valued functions on  $X$  equipped with the sup-norm topology and by  $C(X)^+$  the subset of  $C(X)$  consisting of all nonnegative functions; the supremum norm of  $f \in C(X)$ , namely  $\sup\{|f(x)| | x \in X\}$ , will be denoted by  $\|f\|$ . By an integral on  $C(X)$  we mean a linear functional on  $C(X)$  which takes nonnegative values on  $C(X)^+$ . For an integral  $\Lambda$  on  $C(X)$  the *support* of  $\Lambda$  is defined to be the subset of  $X$  consisting of all  $x \in X$  such that  $\Lambda(f) > 0$  for any  $f \in C(X)^+$  for which  $f(x) > 0$ ; the support is easily seen to be a closed subset of  $X$ . It can also be verified by a simple point-set topological argument that if  $\Lambda$  is an integral on  $C(X)$  and  $f \in C(X)$  vanishes on the support of  $\Lambda$  then  $\Lambda(f) = 0$ . If  $\Lambda$  is an integral on  $C(X)$ , where  $X$  is a compact metrizable space, and  $X'$  is the support of  $\Lambda$  then there exists a unique integral  $\Lambda'$  on  $C(X')$  such that  $\Lambda'(f|_{X'}) = \Lambda(f)$  for all  $f \in C(X)$ , where  $f|_{X'}$  denotes the restriction of  $f$  to  $X'$ ; this follows from the Tietze-Urysohn extension theorem (cf. [D], (4.5.1)) and the above mentioned property of the support. We note also that the support of  $\Lambda'$  as above is the whole of  $X'$ .

For any homeomorphism  $\phi$  of a compact (metrizable) space  $X$  an integral  $\Lambda$  on  $C(X)$  is said to be  $\phi$ -invariant if  $\Lambda(f \circ \phi) = \Lambda(f)$  for all  $f \in C(X)$ ; clearly the support of a  $\phi$ -invariant integral on  $C(X)$  is a  $\phi$ -invariant (closed) subset of  $X$ .

*Proof of Proposition 1.7.* We fix a dense sequence in  $C(X)$ , say  $f_j, j = 1, 2, \dots$ . Let  $x_0 \in X$ . Given  $f_j$ , for any sequence  $\{m_k\}$  of natural numbers  $m_k^{-1} \sum_{i=0}^{m_k-1} f_j \circ \phi^i(x_0)$  is a bounded sequence and therefore admits a convergent subsequence. Using a standard procedure (finding  $\{m_k^{(j)}\}$ , with each sequence a subsequence of the previous one, such that the corresponding sequence for  $f_j$  as above converges and considering  $\{m_k^{(k)}\}$ ) we get a sequence  $\{n_k\}$  of natural numbers such that  $n_k^{-1} \sum_{i=0}^{n_k-1} f_j \circ \phi^i(x_0)$  converges for all  $j$ ; also, the limit is between  $-\|f_j\|$  and  $\|f_j\|$ . Since  $\{f_j\}$  is dense

in  $C(X)$  this readily implies that  $n_k^{-1} \sum_{i=0}^{n_k-1} f \circ \phi^i(x_0)$  converges for all  $f \in C(X)$ ; let  $c_f$  be the limit corresponding to  $f$ . Then it can be verified that  $\Lambda: C(X) \rightarrow \mathbf{R}$  defined by  $\Lambda(f) = c_f$ , for all  $f \in C(X)$ , is a  $\phi$ -invariant integral on  $C(X)$ . Also clearly  $\Lambda$  is not identically zero and therefore by our observations above, the support, say  $X'$ , is a nonempty closed  $\phi$ -invariant subset of  $X$  and further  $C(X')$  admits an integral with full support (namely  $X'$ ) which is invariant under the restriction of  $\phi$  to  $X'$ . Replacing  $X$  as in the hypothesis by  $X'$  we may without loss of generality assume that  $C(X)$  admits a  $\phi$ -invariant integral whose support is  $X$ ; in the rest of the argument we let  $\Lambda$  be any such integral.

Now suppose that there do not exist any recurrent points for  $\phi$ . Let  $\rho(\cdot, \cdot)$  be the metric on  $X$ . Let  $\theta$  be the function on  $X$  defined by  $\theta(x) = \inf\{\rho(\phi^i(x), x) \mid i = 1, 2, \dots\}$ , for all  $x \in X$ . There being no recurrent points means that  $\theta(x) > 0$  for all  $x \in X$ . For each natural number  $k$  let  $E_k = \{x \in X \mid \theta(x) \geq 1/k\}$ . Then each  $E_k$  is a closed subset of  $X$  and  $X = \bigcup E_k$ . Therefore by the Baire category theorem there exists a  $k$  such that  $E_k$  has an interior point in  $X$ . In particular, there exists an open ball, say  $A$ , of radius at most  $1/3k$  contained in  $E_k$ . The definition of  $E_k$  and the condition on the radius of  $A$  then imply that the sets  $\phi^i(A)$ ,  $i \in \mathbf{Z}$ , are mutually disjoint. Now let  $x \in A$  and let  $f \in C(X)^+$  be such that  $f(x) > 0$  and the support of  $f$  (the closure of the set  $\{y \in X \mid f(y) > 0\}$ ) is contained in  $A$ . For each natural number  $n$  let  $S_n(f) = \sum_{i=0}^{n-1} f \circ \phi^i \in C(X)$ . The disjointness of  $\phi^i(A)$ ,  $i \in \mathbf{Z}$ , implies that, for any  $n$ ,  $\|S_n(f)\| = \|f\|$ . Also, by the  $\phi$ -invariance of  $\Lambda$  we have  $\Lambda(S_n(f)) = n\Lambda(f)$ . Hence  $\Lambda(f) = \Lambda(S_n(f))/n \leq \|S_n(f)\| \Lambda(1_X)/n = \|f\| \Lambda(1_X)/n$  for all  $n$ , where  $1_X$  denotes the constant function with value 1. But this implies that  $\Lambda(f) = 0$  contradicting the assumption that the support of  $\Lambda$  is the whole of  $X$ . This proves the proposition.

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