1. Introduction

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AN EXPOSITION OF POINCARÉ'S POLYHEDRON THEOREM

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1. INTRODUCTION

Poincaré's Theorem is an important, widely used and well-known result. There are a number of expositions in the literature (see [Mas71, Sei75, Mor78, Apa86, Mas88]). However, as far as we know, there is no source which contains a completely satisfying proof which applies to all dimensions and all constant curvature geometries. There is a tendency for unnecessary hypotheses to be included, which are sometimes implied by the other hypotheses and sometimes unnecessarily restrict the range of validity of the theorem.

A feature of this paper is the emphasis on the algorithmic aspects of Poincaré's Theorem. This point of view was first stressed by [Ril83]. Riley's work is restricted to dimensions two and three, where various points become easier to analyze. We want procedures which will tell us whether a given finite set of finite-sided convex polyhedra and face-pairings do or do not give rise to an orbifold, or, equivalently to a tessellation of $\mathbf{X}^n$. Such procedures have been exploited to remarkable effect — readers are referred to the paper by Bob Riley just cited, and to other contributions by him.

Riley's computer programs start with a list of group generators, given numerically, and attempt to find a fundamental domain for the group. The procedure goes through a check on a putative fundamental domain, along the lines explained in this paper. An essential further feature of his programs, and of similar programs by others, is that it incorporates another procedure for improving the guess on the shape of the fundamental domain, if the original guess fails. This second feature is not addressed in this paper.

This paper elaborates notes of lectures given by Epstein in 1992 at Warwick University. We will assume that the reader is familiar with the elementary definitions of euclidean, spherical and hyperbolic spaces and their geodesic

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subspaces. We denote these spaces by $\mathbf{E}^n$, $\mathbf{S}^n$ and $\mathbf{H}^n$. When we want to denote one of these three spaces, without specifying which, we will call it $\mathbf{X}^n$. Note that $\mathbf{E}^1$ and $\mathbf{H}^1$ are isometric to each other and locally isometric to $\mathbf{S}^1$.

DEFINITION 1.1 ($\mathbf{X}$-subspace). An $\mathbf{X}$-subspace of $\mathbf{X}^n$ is a copy of $\mathbf{X}^i$, for some i with $0 \leqslant i \leqslant n$, embedded geodesically. In the case when $\mathbf{X}$ is spherical, every 0-dimensional $\mathbf{X}$-subspace is a copy of $\mathbf{S}^0$ in $\mathbf{S}^n$, embedded as a pair of antipodal points.

When reading this paper, it will be helpful for the reader to understand the concept of a manifold or orbifold modelled on one of these spaces. The reader is referred to [Thu, Thu80] for these definitions. [BP92] is also a useful source of background matter.

Suppose G is a discrete group of isometries of $\mathbf{X}^n$. Then one can fix a point $p \in \mathbf{X}^n$ such that no element of G fixes p, and define the *Dirichlet domain of G with centre p*. This is the set of $x \in \mathbf{X}^n$ such that, for all $g \in G$, $d(x, p) \leqslant d(x, gp)$. The Dirichlet domain is a convex polyhedron (see Definition 2.1). It has a finite number of faces in many important cases (for example if the quotient of $\mathbf{X}^n$ by G is compact), but in general may have an infinite number of faces, even if G is cyclic as in Example 1.3. It is a fundamental domain for the action of G on $\mathbf{X}^n$.

EXAMPLE 1.2. Consider the free abelian group on two generators acting on the plane by translation by (m, n), where m and n are integers. The Dirichlet domain centred on any point is a unit square. If the free abelian group acts by translation but does not have a pair of generators acting by orthogonal translations, then the Dirichlet domain has six sides.

EXAMPLE 1.3 (infinitely many faces). Take the isometry of $\mathbf{E}^3$ which is translation along the z-axis followed by an irrational rotation about the z-axis. Let G be the cyclic infinite group generated by this isometry. The Dirichlet domain centred on any point not on the z-axis has an infinite number of faces.

Each element $g \in G$ gives rise to a hyperplane A_g of $\mathbf{X}^n$, consisting of points which are equidistant from p and $g(p)$. Let F_g be the intersection of the Dirichlet domain with A_g. If F_g has dimension $n-1$, it is a face of the Dirichlet domain, and each face of dimension $n-1$ is equal to F_g for some g. If F_g is a face, so is $F_{g^{-1}}$. The element g sends $F_{g^{-1}}$ to F_g and is called a *face-pairing* of the Dirichlet domain.

Now suppose one is presented with a convex polyhedron and, for each face, an isometry pairing it with another face. Poincaré's Theorem is concerned with the question "Can this be the fundamental domain and face-pairings for the action of a discrete group of isometries?" It turns out that this question can be answered with conditions which are surprisingly simple to check, and the answer is the content of Poincaré's Theorem. If the polyhedron has a finite number of faces, the conditions for Poincaré's Theorem can be checked algorithmically.

Here is another point of view on Poincaré's Theorem. A manifold or orbifold modelled on $\mathbf{X}^n$ can be cut along $(n-1)$-dimensional geodesic subspaces to obtain a single convex polyhedron (the fundamental domain), as in the case of the Dirichlet domain. However, it may sometimes be convenient not to use a single polyhedron. We could for example take an arbitrary fundamental domain which is not necessarily convex (and does not necessarily have geodesic faces). This fundamental domain can be approximated by a union of convex pieces. The more convoluted the fundamental domain, the greater the number of convex pieces we might need for a reasonable approximation. As has been pointed out in [Bow93], in dimensions greater than four a geometrically finite discrete hyperbolic group may have a fundamental domain which needs to be built up from a finite number of finite-sided convex pieces — one such does not suffice.

In [Bow93] one finds an example of a four-dimensional hyperbolic manifold which has a fundamental domain with a finite number of faces, all geodesic, but such that no Dirichlet region has finitely many faces. Probably Bowditch's example has the property that every convex fundamental domain, even if not a Dirichlet domain, has to have an infinite number of faces. Since it is nice to use convex building blocks — for example, they can easily be specified using a finite set of real numbers — we would probably want to decompose the fundamental domain in such a case into a finite number of convex polyhedra.

Now suppose we are given a set of convex polyhedra with face-pairings. The role of Poincaré's Theorem is to determine whether this situation can arise from a manifold or orbifold by cutting along geodesic codimension-one subspaces. In each case there is also the problem of determining the associated (fundamental) group from the combinatorial data presented.

Poincaré's Theorem can be used to construct many interesting examples of groups acting on hyperbolic or euclidean space or on the sphere, and many interesting manifolds and orbifolds modelled on one of these spaces. Readers are referred to [Thu, Thu80] for such examples.

There are several reasons why it is better to use several convex building blocks than only one. Firstly, as we have already pointed out, this is necessary if we are to deal with all geometrically finite groups. Secondly many of the most interesting examples are constructed using more than one piece, for example the two ideal regular hyperbolic tetrahedra used to give a complete hyperbolic structure to the figure-eight complement (see [Thu, Thu80]). Thirdly the hypotheses come up naturally in the proof; if one starts with a single convex piece, the natural inductive proof inexorably leads one to consider glueing together several convex pieces in lower dimensions. Fourthly, it may be convenient to use a non-convex fundamental domain, rather than a convex fundamental domain. The non-convex fundamental domains that arise in practice can be cut into a finite number of convex pieces, making our hypotheses applicable.

One way in which our treatment differs from all previous treatments, is that we do not assume we start with an embedded fundamental domain. Instead the fundamental domain is expressed as the union of convex cells, each of which can be separately embedded, without knowing to begin with that their union can be embedded. For example, suppose we are given three planar wedges of angle $5\pi/6$, $6\pi/7$ and $7\pi/8$ with face-pairings glueing them together. The union of these pieces cannot form a fundamental domain, because their union after glueing cannot be embedded. The point here is whether this non-embeddability or embeddability needs to be checked beforehand. Our proof shows that the usual checks for Poincaré's Theorem, in the case where there is only one convex piece, in any case imply the embeddability of the potential fundamental domain, so no special separate check is necessary. In this case the extra necessary checking is easy, but in a more complicated situation, the algorithm presented here could lead to significant saving of time and complication.

2. Convex polyhedra

Let $\mathbf{X}^n$ be hyperbolic, euclidean or spherical n-dimensional space. A hyperplane (that is, a codimension-one $\mathbf{X}$-subspace) divides $\mathbf{X}^n$ into two components; we will call the closure of either of them a *half-space* in $\mathbf{X}^n$. Any $\mathbf{X}$-subspace is the intersection of hyperplanes, and *vice versa*.

Definition 2.1 (convex polyhedron). A connected subset P of $\mathbf{X}^n$ is called a *convex polyhedron* if it is the intersection of a family $\mathscr{H}$ of half-spaces