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ON THE COHOMOLOGY OF COMPACT LIE GROUPS

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ABSTRACT. We give a new computation of the cohomology of a Lie group that some mathematicians may find to be shorter and more elementary than previous approaches. The main new ingredient is a result of L. Solomon on differential forms invariant under a finite reflection group. The cohomology is shown to have a bi-grading which has several interpretations.

1. INTRODUCTION

Let G be a compact connected Lie group, and let T be a maximal torus in G. We denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{t}$. Let W be the Weyl group of T in G. Then W acts on $\mathfrak{t}$ as a group generated by reflections about the kernels of the roots of $\mathfrak{t}$ in $\mathfrak{g} \otimes \mathbf{C}$. It has been known since the first half of this century that the cohomology ring $H(G)$, with real coefficients, is an exterior algebra with generators in degrees $2m_1 + 1, \ldots, 2m_l + 1$, where $m_1 + 1, \ldots, m_l + 1$ are the degrees of homogeneous generators of the ring of W-invariant polynomial functions on $\mathfrak{t}$. In particular, the Poincaré polynomial of G is $(1 + t^{2m_1+1}) \cdots (1 + t^{2m_l+1})$, and G has the cohomology of a product of odd-dimensional spheres.

Despite its age and familiarity, it is not easy to find a proof of this theorem in the literature. There are many beginnings and sketches in the textbooks, but the difficult part, namely the remarkable connection between degrees of invariant polynomials and Betti numbers, usually goes unproven. One reason is that the standard proofs (for example, [Bo2], [Ch], [L]) require substantial algebraic preliminaries on Hopf algebras, spectral sequences, and differential algebras. (See [Bo1] and [Sam] for historical surveys.)

We offer here a new but less sophisticated computation of the cohomology of a Lie group, avoiding the above algebraic techniques. Instead we use

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