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2. Basic material

For more details in this section see [A], for example.

(2.1) Recall that G is a compact connected Lie group with maximal torus T, having respective Lie algebras $\mathfrak{g}$ and $\mathfrak{t}$. The Weyl group is the finite group $W = N/T$, where N is the normalizer in G of T. Since G is compact, there is an $Ad(G)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, obtained by averaging any inner product over G. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to this inner product, so

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \qquad \text{(orthogonal)} .$$

The infinitesimal version of invariance of the inner product is the identity

$$\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0 ,$$

for $X, Y, Z \in \mathfrak{g}$.

(2.2) The exponential map $\exp\colon \mathfrak{g} \to G$ is surjective, since G is compact. This is one of the deeper theorems in a first course on Lie groups. We actually only need this surjectivity for $\exp\colon \mathfrak{t} \to T$, which is clear.

The Lie algebra $\mathfrak{t}$ is abelian (the bracket is zero); in fact $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$. In particular, no nonzero vector in $\mathfrak{m}$ has zero bracket with all of $\mathfrak{t}$. Likewise, $Ad(T)$ has no nonzero invariant vectors in $\mathfrak{m}$.

Now a torus is a topologically cyclic group. That means there exists a *generic element* $t_0 \in T$ whose powers form a dense subgroup of T. It follows that the single operator $Ad(t_0)$ can have no invariants in $\mathfrak{m}$. Likewise in the group G, it can be shown that a maximal torus is its own centralizer, so the centralizer in G of t_0 is just T. There is a similar notion in the Lie algebra. A *regular element* of $\mathfrak{t}$ is one whose $Ad(G)$-centralizer is exactly $Ad(T)$. To find one, take any $H_0 \in \mathfrak{t}$ such that $\exp H_0 = t_0$.

(2.3) The group G acts on $\mathfrak{g}$ via Ad, and this induces an action of W on $\mathfrak{t}$. No element of W acts trivially, and the image of W in $GL(\mathfrak{t})$ is generated by reflections about certain hyperplanes defined as follows.

Since the nontrivial irreducible representations of a torus are given by two dimensional rotations, we have an orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\nu$, where each $\mathfrak{m}_k$ is two dimensional and there is a finite set of nonzero linear functionals $\Delta^+ = \{\alpha_1, \ldots, \alpha_\nu\} \subset \mathfrak{t}^*$, called *positive roots* such that for $H \in \mathfrak{t}$, the eigenvalues of $Ad \exp H$ on $\mathfrak{m}_i$ are $\exp(\pm\sqrt{-1}\,\alpha_i(H))$. We determine the signs as follows. Fix a regular element $H_0 \in \mathfrak{t}$. We may and shall choose the positive roots so that they take strictly positive values on H_0. The action of W on $\mathfrak{t}$ is generated by reflections about the kernels of the positive roots.

Since each $\mathfrak{m}_i$ is also preserved by $ad(\mathfrak{t})$, we can choose an orthonormal basis $\{X_i, X_{\nu+i}\}$ of $\mathfrak{m}_i$ such that, for $H \in \mathfrak{t}$, the matrix of $ad(H)|_{\mathfrak{m}_i}$ with respect to this basis is

$$\begin{pmatrix} 0 & \alpha(H) \\ -\alpha(H) & 0 \end{pmatrix}.$$

Note that the ad-invariance of the inner product $\langle\ ,\ \rangle$ implies, for all $1 \leqslant i \leqslant \nu$, all $1 \leqslant j \leqslant 2\nu$ and all $H \in \mathfrak{t}$ that

$$\langle H, [X_i, X_j]\rangle = \langle [H, X_i], X_j\rangle = -\alpha_i(H)\langle X_{i+\nu}, X_j\rangle .$$

By orthonormality, this last pairing can only be nontrivial if $j = i + \nu$. Hence if $j \neq i + \nu$, we have $[X_i, X_j] \in \mathfrak{m}$. The same thing happens if $i > \nu$ and $j \neq i - \nu$.

On the other hand, for $1 \leqslant i \leqslant \nu$, set $H_i = [X_i, X_{\nu+i}]$. This is $Ad(T)$-invariant, so $H_i \in \mathfrak{t}$, and $ad(H_i)\mathfrak{m}_i \subseteq \mathfrak{m}_i$. It follows that the span of $X_i, X_{i+\nu}, H_i$ is a Lie subalgebra $\mathfrak{g}_i$ of $\mathfrak{g}$. It is always isomorphic to $\mathfrak{su}(2)$.

3. Invariant Theory

All proofs missing from this section may be found in the textbook [H], the expository article [F], or [Bk].

(3.1) Let

$$\mathscr{S} = \bigoplus_{p=0}^{\infty} \mathscr{S}^p \quad \text{and} \quad \Lambda = \bigoplus_{q=0}^{l} \quad (l = \dim \mathfrak{t})$$

be the symmetric and exterior algebras on $\mathfrak{t}^*$, respectively. The adjoint action of W on $\mathfrak{t}$ induces representations of W on $\mathscr{S}$ and Λ by degree-preserving algebra automorphisms. For example, the action of W on Λ^l is multiplication by the *sign character*

$$\varepsilon : W \to \{\pm 1\} \quad \text{given by} \quad \varepsilon(w) = \det Ad(w)_{\mathfrak{t}} .$$

Note that $\varepsilon(w)$ is the parity of the number of reflections needed to express $Ad(w)_{\mathfrak{t}}$.