

## 2. Basic material

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## 2. BASIC MATERIAL

For more details in this section see [A], for example.

(2.1) Recall that  $G$  is a compact connected Lie group with maximal torus  $T$ , having respective Lie algebras  $\mathfrak{g}$  and  $\mathfrak{t}$ . The Weyl group is the finite group  $W = N/T$ , where  $N$  is the normalizer in  $G$  of  $T$ . Since  $G$  is compact, there is an  $Ad(G)$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ , obtained by averaging any inner product over  $G$ . Let  $\mathfrak{m}$  be the orthogonal complement of  $\mathfrak{t}$  in  $\mathfrak{g}$  with respect to this inner product, so

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \quad (\text{orthogonal}) .$$

The infinitesimal version of invariance of the inner product is the identity

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 ,$$

for  $X, Y, Z \in \mathfrak{g}$ .

(2.2) The exponential map  $\exp: \mathfrak{g} \rightarrow G$  is surjective, since  $G$  is compact. This is one of the deeper theorems in a first course on Lie groups. We actually only need this surjectivity for  $\exp: \mathfrak{t} \rightarrow T$ , which is clear.

The Lie algebra  $\mathfrak{t}$  is abelian (the bracket is zero); in fact  $\mathfrak{t}$  is a maximal abelian subalgebra of  $\mathfrak{g}$ . In particular, no nonzero vector in  $\mathfrak{m}$  has zero bracket with all of  $\mathfrak{t}$ . Likewise,  $Ad(T)$  has no nonzero invariant vectors in  $\mathfrak{m}$ .

Now a torus is a topologically cyclic group. That means there exists a *generic element*  $t_0 \in T$  whose powers form a dense subgroup of  $T$ . It follows that the single operator  $Ad(t_0)$  can have no invariants in  $\mathfrak{m}$ . Likewise in the group  $G$ , it can be shown that a maximal torus is its own centralizer, so the centralizer in  $G$  of  $t_0$  is just  $T$ . There is a similar notion in the Lie algebra. A *regular element* of  $\mathfrak{t}$  is one whose  $Ad(G)$ -centralizer is exactly  $Ad(T)$ . To find one, take any  $H_0 \in \mathfrak{t}$  such that  $\exp H_0 = t_0$ .

(2.3) The group  $G$  acts on  $\mathfrak{g}$  via  $Ad$ , and this induces an action of  $W$  on  $\mathfrak{t}$ . No element of  $W$  acts trivially, and the image of  $W$  in  $GL(\mathfrak{t})$  is generated by reflections about certain hyperplanes defined as follows.

Since the nontrivial irreducible representations of a torus are given by two dimensional rotations, we have an orthogonal decomposition  $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\nu$ , where each  $\mathfrak{m}_k$  is two dimensional and there is a finite set of nonzero linear functionals  $\Delta^+ = \{\alpha_1, \dots, \alpha_\nu\} \subset \mathfrak{t}^*$ , called *positive roots* such that for  $H \in \mathfrak{t}$ , the eigenvalues of  $Ad \exp H$  on  $\mathfrak{m}_i$  are  $\exp(\pm \sqrt{-1} \alpha_i(H))$ . We determine the signs as follows. Fix a regular

element  $H_0 \in \mathfrak{t}$ . We may and shall choose the positive roots so that they take strictly positive values on  $H_0$ . The action of  $W$  on  $\mathfrak{t}$  is generated by reflections about the kernels of the positive roots.

Since each  $\mathfrak{m}_i$  is also preserved by  $ad(\mathfrak{t})$ , we can choose an orthonormal basis  $\{X_i, X_{v+i}\}$  of  $\mathfrak{m}_i$  such that, for  $H \in \mathfrak{t}$ , the matrix of  $ad(H)|_{\mathfrak{m}_i}$  with respect to this basis is

$$\begin{pmatrix} 0 & \alpha(H) \\ -\alpha(H) & 0 \end{pmatrix}.$$

Note that the  $ad$ -invariance of the inner product  $\langle \cdot, \cdot \rangle$  implies, for all  $1 \leq i \leq v$ , all  $1 \leq j \leq 2v$  and all  $H \in \mathfrak{t}$  that

$$\langle H, [X_i, X_j] \rangle = \langle [H, X_i], X_j \rangle = -\alpha_i(H) \langle X_{i+v}, X_j \rangle.$$

By orthonormality, this last pairing can only be nontrivial if  $j = i + v$ . Hence if  $j \neq i + v$ , we have  $[X_i, X_j] \in \mathfrak{m}$ . The same thing happens if  $i > v$  and  $j \neq i - v$ .

On the other hand, for  $1 \leq i \leq v$ , set  $H_i = [X_i, X_{v+i}]$ . This is  $Ad(T)$ -invariant, so  $H_i \in \mathfrak{t}$ , and  $ad(H_i)\mathfrak{m}_i \subseteq \mathfrak{m}_i$ . It follows that the span of  $X_i, X_{i+v}, H_i$  is a Lie subalgebra  $\mathfrak{g}_i$  of  $\mathfrak{g}$ . It is always isomorphic to  $\mathfrak{su}(2)$ .

### 3. INVARIANT THEORY

All proofs missing from this section may be found in the textbook [H], the expository article [F], or [Bk].

(3.1) Let

$$\mathcal{S} = \bigoplus_{p=0}^{\infty} \mathcal{S}^p \quad \text{and} \quad \Lambda = \bigoplus_{q=0}^l \Lambda^q \quad (l = \dim \mathfrak{t})$$

be the symmetric and exterior algebras on  $\mathfrak{t}^*$ , respectively. The adjoint action of  $W$  on  $\mathfrak{t}$  induces representations of  $W$  on  $\mathcal{S}$  and  $\Lambda$  by degree-preserving algebra automorphisms. For example, the action of  $W$  on  $\Lambda^l$  is multiplication by the *sign character*

$$\varepsilon: W \rightarrow \{\pm 1\} \quad \text{given by} \quad \varepsilon(w) = \det Ad(w)_{\mathfrak{t}}.$$

Note that  $\varepsilon(w)$  is the parity of the number of reflections needed to express  $Ad(w)_{\mathfrak{t}}$ .