3. Invariant Theory

element $H_0 \in \mathfrak{t}$. We may and shall choose the positive roots so that they take strictly positive values on H_0. The action of W on $\mathfrak{t}$ is generated by reflections about the kernels of the positive roots.

Since each $\mathfrak{m}_i$ is also preserved by $ad(\mathfrak{t})$, we can choose an orthonormal basis $\{X_i, X_{\nu+i}\}$ of $\mathfrak{m}_i$ such that, for $H \in \mathfrak{t}$, the matrix of $ad(H)|_{\mathfrak{m}_i}$ with respect to this basis is

$$\begin{pmatrix} 0 & \alpha(H) \\ -\alpha(H) & 0 \end{pmatrix}.$$

Note that the ad-invariance of the inner product $\langle\ ,\ \rangle$ implies, for all $1 \leqslant i \leqslant \nu$, all $1 \leqslant j \leqslant 2\nu$ and all $H \in \mathfrak{t}$ that

$$\langle H, [X_i, X_j] \rangle = \langle [H, X_i], X_j \rangle = -\alpha_i(H) \langle X_{i+\nu}, X_j \rangle .$$

By orthonormality, this last pairing can only be nontrivial if $j = i + \nu$. Hence if $j \neq i + \nu$, we have $[X_i, X_j] \in \mathfrak{m}$. The same thing happens if $i > \nu$ and $j \neq i - \nu$.

On the other hand, for $1 \leqslant i \leqslant \nu$, set $H_i = [X_i, X_{\nu+i}]$. This is $Ad(T)$-invariant, so $H_i \in \mathfrak{t}$, and $ad(H_i)\mathfrak{m}_i \subseteq \mathfrak{m}_i$. It follows that the span of $X_i, X_{i+\nu}, H_i$ is a Lie subalgebra $\mathfrak{g}_i$ of $\mathfrak{g}$. It is always isomorphic to $\mathfrak{su}(2)$.

3. Invariant Theory

All proofs missing from this section may be found in the textbook [H], the expository article [F], or [Bk].

(3.1) Let

$$\mathscr{S} = \bigoplus_{p=0}^{\infty} \mathscr{S}^p \quad \text{and} \quad \Lambda = \bigoplus_{q=0}^{l} \quad (l = \dim \mathfrak{t})$$

be the symmetric and exterior algebras on $\mathfrak{t}^*$, respectively. The adjoint action of W on $\mathfrak{t}$ induces representations of W on $\mathscr{S}$ and Λ by degree-preserving algebra automorphisms. For example, the action of W on Λ^l is multiplication by the *sign character*

$$\varepsilon \colon W \to \{\pm 1\} \quad \text{given by} \quad \varepsilon(w) = \det Ad(w)_{\mathfrak{t}} .$$

Note that $\varepsilon(w)$ is the parity of the number of reflections needed to express $Ad(w)_{\mathfrak{t}}$.

We are interested in W-invariant polynomials, and more generally, W-invariant differential forms with polynomial coefficients. For the unitary group $U(n)$, the ring of invariants $\mathcal{S}^W$ is generated by the elementary symmetric polynomials $s_1, \ldots, s_n$ in variables $x_1, \ldots, x_n$ defined as

$$s_d(x_1, \ldots, x_n) = \sum_{1 \leqslant i_1 < \cdots < i_d \leqslant n} x_{i_1} \cdots x_{i_d} .$$

The elementary symmetric polynomials are algebraically independent, and their number equals the dimension n of a maximal torus of $U(n)$. In general, we have

(3.2) THEOREM (Chevalley). *The ring* $\mathcal{S}^W$ *has algebraically independent homogeneous generators* $F_1, \ldots, F_l$, *hence is a polynomial ring*

$$\mathcal{S}^W = \mathbf{R}[F_1, \ldots, F_l] .$$

We number these generators so that $\deg F_1 \leqslant \deg F_2 \leqslant \cdots \leqslant \deg F_l$. (Note to experts: Since we are not assuming G to be semisimple, some of the F_i's could have degree one.) The *exponents* $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_l$ of W acting on $\mathfrak{t}$ are defined by the relations $m_i + 1 = \deg F_i$. It is known that $m_1 + \cdots + m_l = \nu$, and $(1 + m_1) \cdots (1 + m_l) = |W|$.

Every compact connected Lie group is, up to finite covering, the product of a central torus with a direct product of classical groups $SU(n)$, $SO(n)$, $Sp(n)$, and exceptional groups G_2, F_4, E_6, E_7, E_8. For these groups the m_i's are given as follows:

$SU(n)$: $1, 2, \ldots, n-1$. $\quad SO(2n)$: $1, 3, \ldots, 2n-3, n-1$.

$SO(2n+1)$ and $Sp(n)$: $1, 3, \ldots, 2n-1$.

G_2: $1, 5$. $\quad F_4$: $1, 5, 7, 11$.

E_6: $1, 4, 5, 7, 8, 11$.

E_7: $1, 5, 7, 9, 11, 13, 17$.

E_8: $1, 7, 11, 13, 17, 19, 23, 29$.

These numbers are easy to verify for the classical groups and G_2 (whose maximal torus T is that of $SU(3)$ with Weyl group S_3 extended by the inverse map on T), using elementary symmetric polynomials as above. Computing the exponents for the other exceptional groups is more difficult. See [C].

(3.3) The W-module structure of the whole polynomial ring $\mathcal{S}$ is given as follows. Let $\mathcal{D}$ be the ring of constant coefficient differential operators on $\mathcal{S}$. We can think of $\mathcal{D}$ as the symmetric algebra $S(\mathfrak{t})$, where $H \in \mathfrak{t}$

corresponds to the derivation of $\mathscr{S}$ extending the functional on $\mathfrak{t}^*$ given by evaluation at H. Then W acts naturally on $\mathscr{D}$ and one defines the "harmonic polynomials" in $\mathscr{S}$ to be those annihilated by the W-invariant differential operators:

$$\mathscr{H} = \{ f \in \mathscr{S} \colon \mathscr{D}^W f = 0 \} .$$

Let $\mathscr{H}^p = \mathscr{H} \cap \mathscr{S}^p$. Then $\mathscr{H} = \oplus_p \mathscr{H}^p$, since a differential operator is W-invariant only if each of its homogeneous components is so. The action of W on $\mathscr{S}$ leaves $\mathscr{H}$ invariant.

Let $\mathscr{I}$ be the ideal in $\mathscr{S}$ generated by the elements of $\mathscr{S}^W$ of positive degree. It is known (see [H, p. 360] that $\mathscr{S} = \mathscr{H} \oplus \mathscr{I}$, and the multiplication map is a linear isomorphism $\mathscr{H} \otimes \mathscr{S}^W \xrightarrow{\simeq} \mathscr{S}$. The former implies that $\mathscr{S}/\mathscr{I}$ and $\mathscr{H}$ are isomorphic W-modules. They are in fact isomorphic to the regular representation of W, as we shall see in (5.4). The isomorphism $\mathscr{H} \otimes \mathscr{S}^W \simeq \mathscr{S}$ implies the identity

$$\sum_{p \geqslant 0} \dim \mathscr{H}^p t^p = \prod_{i=1}^{l} (1 + t + t^2 + \cdots + t^{m_i}) ,$$

which in turn shows that $\dim \mathscr{H}^\nu = 1$, and $\mathscr{H}^p = 0$ for $p > \nu$.

(3.4) Let V be any irreducible W-module. Suppose V is a constituent of $\mathscr{S}^b$, and not a constituent of $\mathscr{S}^c$, for any $c < b$. We call b the *birthday* of V. Then the V-isotypic component of $\mathscr{S}^b$ must consist of harmonic polynomials, for otherwise, a W-invariant differential operator of positive degree would intertwine V with a space of polynomials of lower degree.

For example, the *primordial* harmonic polynomial is

$$\Pi = \prod_{\alpha \in \Delta^+} \alpha \in \mathscr{H}^\nu ,$$

where we recall that Δ^+ is the set of positive roots. For $U(n)$, Π is the van der Monde determinant $\prod_{i<j} x_i - x_j$, which transforms under the symmetric group S_n by the sign character. In general, Π transforms by the sign character ε of W, and any other polynomial transforming by ε must vanish on all root hyperplanes, hence be divisible by Π. Therefore Π is harmonic, ν is the birthday of ε and (1.4) shows that $\mathscr{H}^\nu$ is spanned by Π.

We say that Π is primordial because $\mathscr{H}$ is spanned by the partial derivatives of Π (see [S]). This turns out to be the algebraic analogue of Poincaré duality for G/T.

As we have seen, the sign character is also afforded by Λ^l. In general, if $\mathfrak{g}$ is simple then each exterior power Λ^q is an irreducible W-module. We shall determine the birthday of each Λ^q shortly.

(3.5) Now consider the algebra $\mathcal{S} \otimes \Lambda$ of differential forms on $\mathfrak{t}$ with polynomial coefficients. Let $F_1, \ldots, F_l$ be homogeneous generators of $\mathcal{S}^W$ as in (3.2). Extending that result, Solomon [Sol] has described the W-invariants in $\mathcal{S} \otimes \Lambda$. Because it seems not so well known but is important here, we give a proof, taken from [H].

(3.6) THEOREM (Solomon). *The space* $(\mathcal{S} \otimes \Lambda)^W$ *of* W*-invariants in* $\mathcal{S} \otimes \Lambda$ *is a free* $\mathcal{S}^W$*-module with basis*

$$\{dF_{i_1} \wedge \cdots \wedge dF_{i_q} : 1 \leqslant i_1 < \cdots < i_q \leqslant l\}\,.$$

Proof. It is a general fact about polynomials that the algebraic independence of $F_1, \ldots, F_l$ is equivalent to the form $dF_1 \wedge \cdots \wedge dF_l$ not being identically zero. Let $x_1, \ldots, x_l$ be a basis of $\mathfrak{t}^*$. Then

$$dF_1 \wedge \cdots \wedge dF_l = J dx_1 \cdots dx_l\,,$$

where the Jacobian J is a polynomial of degree $m_1 + \cdots + m_l = \nu$. The left side is W-invariant and $dx_1 \wedge \cdots \wedge dx_l$ affords the sign character ε. Hence J must also afford ε and, because of its degree, J must be a nonzero multiple of the primordial harmonic polynomial Π. Thus

$$dF_1 \wedge \cdots \wedge dF_l = c\Pi dx_1 \wedge \cdots \wedge dx_l\,,$$

for some nonzero real number c.

For a sequence $I = i_1 < \cdots < i_q$, let I' be the increasing sequence of all integers in $\{1, \ldots, l\} - \{i_1, \ldots, i_q\}$. Set $dF_I = dF_{i_1} \wedge \cdots \wedge dF_{i_q}$ for any sequence I. Let k be the quotient field of $\mathcal{S}$. If $f_I \in k$ are such that $\sum_I f_I dF_I = 0$ then multiplying by $dF_{I'}$ kills all terms but I, leaving $\pm c f_I \Pi dx_1 \cdots dx_l = 0$, so $f_I = 0$. Counting dimensions, we find that the dF_I are a k-basis of $k \otimes \Lambda$, and are in particular linearly independent over $\mathcal{S}^W$. Now suppose $\omega \in \mathcal{S} \otimes \Lambda$ is W-invariant. We can express $\omega = \sum_I g_I dF_I$ for some $g_I \in k$. Multiplying by $dF_{I'}$ again, we have

$$\omega \wedge dF_{I'} = \pm c g_I \Pi dx_1 \cdots dx_l \in [\mathcal{S} \otimes \Lambda]^W\,.$$

This forces g_I to be not only W-invariant, but also polynomial. $\square$

For $\omega \in \mathcal{S} \otimes \Lambda$, let $\omega' \in \mathcal{S}/\mathcal{I} \otimes \Lambda$ be obtained by reducing the coefficients of ω modulo $\mathcal{I}$. This induces an exact sequence

$$0 \to (\mathcal{I} \otimes \Lambda)^W \to (\mathcal{S} \otimes \Lambda)^W \xrightarrow{\omega \mapsto \omega'} (\mathcal{S}/\mathcal{I} \otimes \Lambda)^W \to 0\,.$$

It follows immediately from Solomon's theorem that $\{dF'_{i_1} \wedge \cdots \wedge dF'_{i_q} : 1 \leqslant i_1 < \cdots < i_q \leqslant l\}$ spans $(\mathcal{S}/\mathcal{I} \otimes \Lambda)^W$ (over $\mathbf{R}$). This is in fact a basis, since $\mathscr{S}/\mathscr{I}$ affords the regular representation of W, so $\dim(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W = 2^l$. We therefore have the following

(3.7) COROLLARY. *$(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W$ is an exterior algebra with generators*

$$dF_i' \in [(\mathscr{S}/\mathscr{I})^{m_i} \otimes \Lambda^1]^W, \quad \textit{for } 1 \leqslant i \leqslant l .$$

We will see later that this exterior algebra is, with degrees in $\mathscr{S}/\mathscr{I}$ doubled, the cohomology ring of the compact Lie group G. As W-representations, we have $\mathscr{S}/\mathscr{I} \simeq \mathscr{H}$ and the corollary gives the following

(3.8) MULTIPLICITY FORMULA.

$$\sum_{n=0}^{\nu} \dim \mathrm{Hom}_W(\Lambda^q, \mathscr{H}^n) u^n = s_q(u^{m_1}, \ldots, u^{m_l}) ,$$

where s_q is the elementary symmetric polynomial in l-variables, and the m_i are the exponents of W.

In particular, the birthday of Λ^q is $m_1 + \cdots + m_q$, if $\mathfrak{g}$ is simple.

(3.9) We close this section with a digression. Suppose $\mathfrak{g}$ is simple, so all Λ^q are irreducible W-modules. We can actually witness the birth of Λ^q in $\mathscr{H}$ using the differentials dF_i, as follows. Choose a basis $x_1, \ldots, x_l$ of $\mathfrak{t}^*$, and consider a q-form

$$\omega = \sum f_{i_1, \ldots, i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_q} \in \mathscr{S} \otimes \Lambda^q .$$

The linear span of the coefficient polynomials $f_{i_1, \ldots, i_q}$ is independent of the choice of basis $\{x_i\}$. Moreover, if ω is W-invariant and nonzero, then its coefficients span a W-invariant subspace of $\mathscr{S}$ which is isomorphic to Λ^q as a W-module, since the latter is irreducible and self-contragredient.

For example, we have seen that

$$dF_1 \wedge \cdots \wedge dF_l = c\Pi dx_1 \wedge \cdots \wedge dx_q ,$$

where c is a nonzero scalar, and Π is the primordial harmonic polynomial, affording the sign character of W. We have a generalization of this for all Λ^q.

(3.10) PROPOSITION. *For $1 \leqslant q \leqslant l$, the coefficients of $dF_1 \wedge \cdots \wedge dF_q$ are harmonic polynomials. They span an irreducible W-submodule of $\mathscr{H}^{m_1 + \cdots + m_q}$, isomorphic to Λ^q.*

Proof. The coefficients of $dF_1 \wedge \cdots \wedge dF_q \in (S^{m_1 + \cdots + m_q} \otimes \Lambda^q)^W$ span a W-invariant subspace of $S^{m_1 + \cdots + m_q}$, isomorphic to Λ^q. As in (3.4), these coefficients are harmonic because $m_1 + \cdots + m_q$ is the birthday of Λ^q, by the multiplicity formula (3.8). □

4. Invariant Differential Forms

The ideas in this section go back to E. Cartan and de Rham. For a thorough exposition, see [C-E].

(4.1) Suppose a compact Lie group G acts transitively on a manifold M. Let τ_g be the diffeomorphism of M corresponding to $g \in G$. A differential p-form $\omega \in \Omega^p(M)$ is *G-invariant* if $\tau_g^* \omega = \omega$. Such a form is determined by its value at any one point of M. One shows by averaging that every de Rham cohomology class on M is represented by a G-invariant form, and that the subcomplex of invariant forms is preserved by the exterior derivative.

Identify $M = G/K$ where K is the stabilizer of a point $o \in M$. We have an orthogonal decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$, where $\mathfrak{r}$ is the Lie algebra of K. Moreover this decomposition is preserved by $Ad(K)$. For example if G acts on itself by left multiplication then $K = 1$ and $\mathfrak{n} = \mathfrak{g}$. For another example take $M = G/T$, so $K = T$ and $\mathfrak{n} = \mathfrak{m}$. In general, $\mathfrak{n}$ is naturally identified with the tangent space $T_o(M)$, so an invariant form $\tilde{\omega}$ is determined by the skew-symmetric multilinear map

$$\omega = \tilde{\omega}_o : \mathfrak{n} \times \cdots \times \mathfrak{n} \to \mathbf{R} .$$

That is, $\omega \in \Lambda^p \mathfrak{n}^*$. The invariance of $\tilde{\omega}$ under K implies the $Ad(K)$-invariance of ω. Conversely, any element $\omega \in (\Lambda^p \mathfrak{n}^*)^K$ determines a G-invariant form $\tilde{\omega}$ on M by the formula

$$\tilde{\omega}_{g \cdot o}\big((d\tau_g)_o X_1, \ldots, (d\tau_g)_o X_p\big) = \omega(X_1, \ldots, X_p) ,$$

for $X_1, \ldots, X_p \in \mathfrak{n}$ and $g \in G$. Thus we may identify the G-invariant p-forms on M with the space $(\Lambda^p \mathfrak{n}^*)^K$. In this view, the exterior derivative becomes the map $\delta : (\Lambda^p \mathfrak{n}^*)^K \to (\Lambda^{p+1} \mathfrak{n}^*)^K$ given by

$$\delta\omega(X_0, \ldots, X_p) = \frac{1}{p+1} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_{\mathfrak{n}}, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) .$$

Here $\hat{}$ means the term is omitted, and $[X_i, X_j]_{\mathfrak{n}}$ is the projection of $[X_i, X_j]$ into $\mathfrak{n}$ along $\mathfrak{r}$. The complex $\{(\Lambda^p \mathfrak{n}^*)^K, \delta\}$ computes the de Rham cohomology of M.