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Artikel: SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

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Kapitel: 3. The Poincaré-Tanaka and Chern-Ji theorems

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let $G = \{e, \tau\} \cdot \mathrm{PGL}(n+1, \mathbf{C})$, where $\tau : \mathbf{P}^n_{\mathbf{C}} \to \mathbf{P}^n_{\mathbf{C}}$ is given by $\tau(z) = \bar{z}$ and e is the identity map. By Lemmas 5 and 4 applied to the restrictions $f|_{U_j}$, there are transformations $A_j \in G$ such that $f|_{U_j} = A_j|_{U_j}$. Since an element of G is uniquely determined by its values on a nonempty open subset of $\mathbf{P}^n_K$ and $(U_1 \cup \cdots \cup U_j) \cap U_{j+1} \neq \emptyset$, it follows by induction that $A_j = A_1$ for all j. Hence $f = A_1|_U$. □

3. The Poincaré-Tanaka and Chern-Ji theorems

The Segre family $\mathscr{M}_{B_n}$ mentioned in the introduction has the projective analogue

$$\mathscr{M}^n_K = \{(z, w) \in \mathbf{P}^n_K \times \mathbf{P}^n_K : \sum_{j=0}^{n} z_j w_j = 0\} .$$

(In fact $\mathscr{M}^n_K$ is a compactification of $\mathscr{M}_{B_n}$; see the proof of Corollary 8.) We let $\pi_i : \mathbf{P}^n_K \times \mathbf{P}^n_K \to \mathbf{P}^n_K$ denote the projection to the i-th factor, for $i = 1, 2$. The main result of this section is the following generalization of the Chern-Ji theorem [CJ, Theorem 2]; our generalization says that a pair of local homeomorphisms of $\mathbf{P}^n_K$ ($K = \mathbf{R}$ or $\mathbf{C}$) mapping $\mathscr{M}^n_K$ into itself must be projective-linear, or possibly anti-projective-linear (if $K = \mathbf{C}$):

THEOREM 6. *Let* $(a^1, a^2) \in \mathscr{M}^n_K$, *where* $K = \mathbf{R}$ *or* $\mathbf{C}$, $n \geqslant 2$. *Let* U_1, U_2 *be open sets in* $\mathbf{P}^n_K$ *containing* a^1, a^2 *respectively, and let* V_i *be the connected component of* $\pi_i(\mathscr{M}^n_K \cap U_1 \times U_2)$ *containing* a_i, *for* $i = 1, 2$. *If* $f_i : U_i \to \mathbf{P}^n_K$ $(i = 1, 2)$ *are continuous injective maps such that*

$$(f_1 \times f_2)(\mathscr{M}^n_K \cap U_1 \times U_2) \subset \mathscr{M}^n_K ,$$

then there exists $A \in \mathrm{PGL}(n+1, K)$ *such that*

(i) $f_1 = A$ *on* V_1 *and* $f_2 = {}^tA^{-1}$ *on* V_2, *if* $K = \mathbf{R}$,

(ii) *either* (i) *holds or* $\bar{f}_1 = A$ *on* V_1 *and* $\bar{f}_2 = {}^tA^{-1}$ *on* V_2, *if* $K = \mathbf{C}$.

REMARK. If the sets $\pi_i(\mathscr{M}^n_K \cap U_1 \times U_2)$ are connected, then $V_i = \pi_i(\mathscr{M}^n_K \cap U_1 \times U_2)$ and we have $\mathscr{M}^n_K \cap U_1 \times U_2 = \mathscr{M}^n_K \cap V_1 \times V_2$. In fact, if we assume that only one of the projections $\pi_1(\mathscr{M}^n_K \cap U_1 \times U_2)$ is connected, then by the uniqueness of A it follows that the conclusion of Theorem 6 holds with $V_i = \pi_i(\mathscr{M}^n_K \cap U_1 \times U_2)$, for $i = 1, 2$.

Proof of Theorem 6. For a point $w \in \mathbf{P}_K^n$ we write

$$w^\perp = \{z \in \mathbf{P}_K^n : z \cdot w = 0\} ,$$

where $z \cdot w = \sum_{j=0}^n z_j w_j$. For a subset $S \subset \mathbf{P}_K^n$ we also write

$$S^\perp = \{z \in \mathbf{P}_K^n : z \cdot w = 0 \ \forall w \in E\} .$$

We consider the collection of lines

$$\mathscr{L}_0 = \{L \in \mathscr{L}(V_1) : L^\perp \cap U_2 \neq \emptyset\} ,$$

which is open in $\mathscr{L}(V_1)$. If z is an arbitrary point of V_1, then by hypothesis we can choose $w \in U_2$ such that $(z, w) \in \mathscr{M}_K^n$. If we let L be any projective line in $w^\perp$ containing z, then $w \in L^\perp \cap U_2$ and hence $L \in \mathscr{L}_0$. Therefore $\bigcup \mathscr{L}_0 \supset V_1$.

Now let $L \in \mathscr{L}_0$ be arbitrary. We claim that we can choose points $w^1, \ldots, w^{n-1} \in L^\perp \cap U_2$, such that $f_2(w^1), \ldots, f_2(w^{n-1})$ are in general position: If $n = 2$, the claim is a tautology, so suppose $n \geqslant 3$. If the claim were false, then $f_2(L^\perp \cap U_2)$ must lie in a projective linear subspace $\mathbf{P}(E)$ of dimension $n-3$ (where E is a linear subspace of K^{n+1} of dimension $n-2$). But then f_2 would be a continuous injection from $(L^\perp \cap U_2)$, which has topological dimension $n-2$ or $2n-4$ (depending on whether K equals $\mathbf{R}$ or $\mathbf{C}$), into $\mathbf{P}(E)$, which has topological dimension $n-3$ or $2n-6$. This contradicts dimension theory.

Let $w^1, \ldots, w^{n-1} \in L^\perp \cap U_2$, such that $f_2(w^1), \ldots, f_2(w^{n-1})$ are in general position, as above. By moving the points slightly if necessary, we can assume also that $w^1, \ldots, w^{n-1}$ are in general position, and hence $L = \langle w^1, \ldots, w^{n-1} \rangle^\perp$. We note that by hypothesis, $f_1(w^\perp \cap U_1) \subset f_2(w)^\perp$ for all $w \in U_2$. Therefore

$$f_1(L \cap U_1) = \bigcap_{j=1}^{n-1} f_1(w^{j\perp} \cap U_1) \subset \bigcap_{j=1}^{n-1} f_2(w^j)^\perp$$

$$= \langle f_2(w^1), \ldots, f_2(w^{n-1}) \rangle^\perp \in \mathscr{L}_K^n(U_1) .$$

Let G be the group of projective-linear, and if $K = \mathbf{C}$, anti-projective linear, transformations of $\mathbf{P}_K^n$ as in the proof of Theorem 3. By Theorem 3, there exists $A \in G$ such that $f_1 = A$ on V_1; similarly, there exists $B \in G$ such that $f_2 = B$ on V_2. By replacing $f_1 \times f_2$ with $\bar{f}_1 \times \bar{f}_2$ if necessary, we can assume that $A \in \mathrm{PGL}(n+1, K)$. We now show that $B = {}^tA^{-1}$: Let M be the connected component of $\mathscr{M}_K^n \cap U_1 \times U_2$ containing (a^1, a^2). Fix a point $w \in \pi_2(M) \subset V_2$, and choose $z^1, \ldots, z^n \in w^\perp \cap V_1$ in general position. Then $(Az^j, Bw) = (f_1(z^j), f_2(w)) \in \mathscr{M}_K^n$ since $(z^j, w) \in \mathscr{M}_K^n$, and thus

$$0 = Az^j \cdot Bw = z^j \cdot {}^tABw\ ,$$

for $j = 1, \dots, n$. Therefore ${}^tABw \in w^{\perp\perp} = \{w\}$. Since w is an arbitrary point of $\pi_2(M)$ and since elements of G are uniquely determined by their values on the open set $\pi_2(M)$, it follows that ${}^tAB$ is the identity $e \in G$, and therefore $B = {}^tA^{-1} \in \mathrm{PGL}(n+1, K)$. □

COROLLARY 7 (Chern-Ji [CJ, Theorem 2]). *Suppose* $U, \hat{U}, V, \hat{V}$ *are connected open sets in* $\mathbf{P}^n_{\mathbf{C}}$ *such that* $\mathcal{M}^n_{\mathbf{C}} \cap U \times V \neq \emptyset$. *If* $f: U \to \hat{U}, g: V \to \hat{V}$ *are biholomorphic maps such that*

$$(f \times g)\,(\mathcal{M}^n_{\mathbf{C}} \cap U \times V) \subset \mathcal{M}^n_{\mathbf{C}}\ ,$$

then f *and* g *are restrictions of elements of* $\mathrm{PGL}(n+1, \mathbf{C})$.

We conclude this paper by demonstrating how the following theorem of Poincaré and Tanaka is obtained from Corollary 7.

COROLLARY 8 (Poincaré-Tanaka Theorem) [Po], [Ta]. *Let* B_n *denote the unit ball in* $\mathbf{C}^n, n \geqslant 2$. *Suppose that* U *is a connected open set in* $\mathbf{C}^n$ *such that* $U \cap \partial B_n \neq \emptyset$. *If* $f: U \to \mathbf{C}^n$ *is a nonconstant holomorphic map such that* $f(U \cap \partial B_n) \subset \partial B_n$, *then* $f|_{U \cap B_n}$ *extends to an automorphism of* B_n.

Proof. By an elementary argument given by H. Alexander ([A], p. 250]), we can assume that the Jacobian matrix of f is nonsingular at some point $z_0 \in U \cap \partial B_n$. (We shall give Alexander's argument later.) By replacing U with a neighborhood of z_0, we can assume that f is injective. Let $\tau: \mathbf{C}^n \to \mathbf{C}^n$ be the conjugation $z \mapsto \bar{z}$. Let $V = \tau(U)$ and consider the holomorphic map $g = \tau \circ f \circ \tau: V \to \mathbf{C}^n$. We let $\hat{U} = f(U)$, $\hat{V} = g(V) = \tau(\hat{U})$ so that the maps $f: U \to \hat{U}$, $g: V \to \hat{V}$ are biholomorphic. We let ψ denote the function on $\mathbf{C}^n \times \mathbf{C}^n$ given by $\psi(z, w) = \sum_{j=1}^n z_j w_j - 1$ and we consider the "Segre family"

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \psi(z, w) = 0\}\ .$$

Let $S: \mathbf{C}^n \to \mathbf{C}^{2n}$ be given by $S(z) = (z, \bar{z})$, so that $S^{-1}(\mathcal{M}_{B_n}) = \partial B_n$ and $S \circ f = (f \times g) \circ S$. Let $\Omega = U \times V$ and $N = S(\partial B_n) = \mathcal{M}_{B_n} \cap S(\mathbf{C}^n)$. Then

$$(f \times g)\,(\Omega \cap N) = S \circ f(U \cap \partial B_n) \subset S(\partial B_n) = N \subset \mathcal{M}_{B_n}\ .$$

Choose a point $z_0 \in U \cap \partial B_n$; then $(z_0, \bar{z}_0) \in \Omega \cap N$. Since $\psi \circ (f \times g)$ vanishes on $\Omega \cap N$ and N is a totally real submanifold of (real) dimension $2n - 1$ in $\mathcal{M}_{B_n}$, it follows that $\psi \circ (f \times g)$ vanishes on the connected component of $\Omega \cap \mathcal{M}_{B_n}$ containing $(z_0, \bar{z}_0)$. After shrinking U if necessary, we can assume that $\psi \circ (f \times g)$ vanishes on $\Omega \cap \mathcal{M}_{B_n}$ and thus $(f \times g)(\Omega \cap \mathcal{M}_{B_n}) \subset \mathcal{M}_{B_n}$. We consider the embedding $\iota \times \iota : \mathbf{C}_n \times \mathbf{C}_n \hookrightarrow \mathbf{P}^n_{\mathbf{C}} \times \mathbf{P}^n_{\mathbf{C}}$ given by $\iota(z_1, \ldots, z_n) = (\sqrt{-1} : z_1 : \ldots : z_n)$, which maps $\mathcal{M}_{B_n}$ onto a (dense open) subset of $\mathcal{M}^n_{\mathbf{C}}$. By Corollary 7 applied to the maps

$$\tilde{f} = \iota \circ f \circ \iota^{-1} : \iota(U) \to \iota(\hat{U}), \quad \tilde{g} = \iota \circ g \circ \iota^{-1} : \iota(V) \to \iota(\hat{V}),$$

there exists $A \in \mathrm{PGL}(n+1, \mathbf{C})$ such that $\tilde{f} = A|_{\iota(U)}$. Thus f extends to the fractional linear map $\iota^{-1} \circ A \circ \iota$, which gives an automorphism of B_n.

We now give a simplified form of Alexander's proof [Al, p. 250] that the Jacobian matrix of the map f must be nonsingular at some point of $U \cap \partial B_n$. We begin by observing that $f^{-1}(\partial B_n)$ is nowhere dense. Indeed, suppose on the contrary that $f^{-1}(\partial B_n)$ contains a connected open set U_0 and assume without loss of generality that $f(z_0) = (1, 0, \ldots, 0)$ for some point $z_0 \in U_0$. Then by the maximum principle, $f_1 \equiv 1$ and hence $f \equiv (1, 0, \ldots, 0)$ on U_0 and thus on U, contradicting the assumption that f is nonconstant. Now suppose on the contrary that the Jacobian determinant of f vanishes identically on $U \cap \partial B_n$. Since the zero of the Jacobian determinant is an analytic subvariety, the Jacobian determinant must vanish identically on U. As a consequence, the fibers of f contain no isolated points. Assume without loss of generality that $(1, 0, \ldots, 0) \in U$ and choose $r < 1$ such that the spherical cap $W := \{z \in B_n : \mathrm{Re}\, z_1 > r\}$ is contained in U. Choose a point $p \in W$ such that $f(p) \notin \partial B_n$. Let A be the connected component of $f^{-1}(f(p)) \cap W$ that contains p; A is an analytic subvariety of W of positive dimension. Furthermore $\bar{A} \backslash A \subset \{z \in \mathbf{C}^n : \mathrm{Re}\, z_1 = r\}$. By the maximum principle (see for example [Gu, Theorem H2]) applied to the holomorphic function $\varphi : A \to \mathbf{C}$ given by $\varphi(z) = \exp z_1$, we conclude that φ is constant and thus $\bar{A} \backslash A = \emptyset$ so that A is a compact subvariety of W of positive dimension, which is impossible. □

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