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Artikel: SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

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sion $2n - 1$ in $\mathscr{M}_{B_n}$, it follows that $\psi \circ (f \times g)$ vanishes on the connected component of $\Omega \cap \mathscr{M}_{B_n}$ containing $(z_0, \bar{z}_0)$. After shrinking U if necessary, we can assume that $\psi \circ (f \times g)$ vanishes on $\Omega \cap \mathscr{M}_{B_n}$ and thus $(f \times g)(\Omega \cap \mathscr{M}_{B_n}) \subset \mathscr{M}_{B_n}$. We consider the embedding $\iota \times \iota : \mathbf{C}_n \times \mathbf{C}_n \hookrightarrow \mathbf{P}^n_{\mathbf{C}} \times \mathbf{P}^n_{\mathbf{C}}$ given by $\iota(z_1, \ldots, z_n) = (\sqrt{-1} : z_1 : \ldots : z_n)$, which maps $\mathscr{M}_{B_n}$ onto a (dense open) subset of $\mathscr{M}^n_{\mathbf{C}}$. By Corollary 7 applied to the maps

$$\tilde{f} = \iota \circ f \circ \iota^{-1} : \iota(U) \to \iota(\hat{U}), \quad \tilde{g} = \iota \circ g \circ \iota^{-1} : \iota(V) \to \iota(\hat{V}),$$

there exists $A \in \mathrm{PGL}(n+1, \mathbf{C})$ such that $\tilde{f} = A\,|_{\iota(U)}$. Thus f extends to the fractional linear map $\iota^{-1} \circ A \circ \iota$, which gives an automorphism of B_n.

We now give a simplified form of Alexander's proof [Al, p. 250] that the Jacobian matrix of the map f must be nonsingular at some point of $U \cap \partial B_n$. We begin by observing that $f^{-1}(\partial B_n)$ is nowhere dense. Indeed, suppose on the contrary that $f^{-1}(\partial B_n)$ contains a connected open set U_0 and assume without loss of generality that $f(z_0) = (1, 0, \ldots, 0)$ for some point $z_0 \in U_0$. Then by the maximum principle, $f_1 \equiv 1$ and hence $f \equiv (1, 0, \ldots, 0)$ on U_0 and thus on U, contradicting the assumption that f is nonconstant. Now suppose on the contrary that the Jacobian determinant of f vanishes identically on $U \cap \partial B_n$. Since the zero of the Jacobian determinant is an analytic subvariety, the Jacobian determinant must vanish identically on U. As a consequence, the fibers of f contain no isolated points. Assume without loss of generality that $(1, 0, \ldots, 0) \in U$ and choose $r < 1$ such that the spherical cap $W := \{z \in B_n : \operatorname{Re} z_1 > r\}$ is contained in U. Choose a point $p \in W$ such that $f(p) \notin \partial B_n$. Let A be the connected component of $f^{-1}(f(p)) \cap W$ that contains p; A is an analytic subvariety of W of positive dimension. Furthermore $\bar{A} \backslash A \subset \{z \in \mathbf{C}^n : \operatorname{Re} z_1 = r\}$. By the maximum principle (see for example [Gu, Theorem H2]) applied to the holomorphic function $\varphi : A \to \mathbf{C}$ given by $\varphi(z) = \exp z_1$, we conclude that φ is constant and thus $\bar{A} \backslash A = \emptyset$ so that A is a compact subvariety of W of positive dimension, which is impossible. □

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