

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 41 (1995)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CUBIC FORMS AND COMPLEX 3-FOLDS
Autor: Okonek, Ch. / Van de Ven, A.
Kapitel: 2.1 COHOMOLOGY RINGS OF 6-MANIFOLDS
DOI: <https://doi.org/10.5169/seals-61829>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 30.01.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

2.1 COHOMOLOGY RINGS OF 6-MANIFOLDS

Let (r, H, w, τ, F, p) be a system of invariants as in section 1; recall that it is admissible iff for every $W \in H$, $T \in H^\vee$ with $\bar{W} = w(\text{mod } 2)$, $\bar{T} \equiv \tau(\text{mod } 2)$ the following congruence holds:

$$(*) \quad W^3 \equiv (p + 24T)(W) \pmod{48}.$$

LEMMA 1. (r, H, w, τ, F, p) is admissible if and only if there exist $W_\circ \in H$, $T_\circ \in H^\vee$ with $\bar{W}_\circ \equiv w(\text{mod } 2)$, $\bar{T}_\circ \equiv \tau(\text{mod } 2)$, such that

- i) $W_\circ^3 \equiv (p + 24T_\circ)(W_\circ) \pmod{48}$
- ii) $p(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2 \pmod{24} \quad \forall x \in H$.

Proof. Obvious since the set of integral lifts of w is a coset $W_\circ + 2H$.

DEFINITION 3. Let $F \in S^3H^\vee$ be a symmetric trilinear form on a finitely generated free abelian group H . An element $W \in H$ is characteristic for F iff

$$(**) \quad x \cdot y \cdot (x + y + W) \equiv 0 \pmod{2} \quad \forall x, y \in H.$$

LEMMA 2. $W \in H$ is a characteristic element for $F \in S^3H^\vee$ if and only if the function $l_W: H \rightarrow \mathbf{Z}$, $l_W(x) := 4x^3 + 6x^2W + 3xW^2$ is linear in x modulo 24.

Proof. $l_W(x + y) = l_W(x) + l_W(y) + 12(x^2y + xy^2 + xyW)$, whence the assertion.

The existence of characteristic elements is a necessary and sufficient condition for a cubic form $F \in S^3H^\vee$ to be realizable by a manifold. In fact, we have:

PROPOSITION 2. A given cubic form $F \in S^3H^\vee$ on a finitely generated free abelian group H is realizable as cup-form of a 1-connected, closed, oriented, 6-dimensional manifold with torsion-free homology if and only if it possesses a characteristic element.

Proof. If (r, H, w, τ, F, p) is an admissible system of invariants, and $W_\circ \in H$ any integral lift of w , then we have $p(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2 \pmod{24} \quad \forall x \in H$, i.e. the function $l_{W_\circ}: H \rightarrow \mathbf{Z}$ is linear modulo 24, and W_\circ is therefore characteristic for F . Conversely, suppose $W_\circ \in H$ is a characteristic element for a cubic form $F \in S^3H^\vee$; let $w := \bar{W}_\circ(\text{mod } 2)$, $r := 0$.

By the main lemma we have to construct linear forms $p, T \in H^\vee$, such that

- i) $W_\circ^3 \equiv (p + 24T)(W_\circ) \pmod{48}$
- ii) $p(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 \pmod{24} \quad \forall x \in H$.

The function $l_{W_\circ} : H \rightarrow \mathbf{Z}, l_{W_\circ}(x) = 4x^3 + 6x^2 W_\circ + 3x W_\circ^2$ is linear modulo 24 since W_\circ is a characteristic element for F : we therefore choose a linear form $p_\circ \in H^\vee$ with $p_\circ(x) \equiv l_{W_\circ}(x) \pmod{24} \quad \forall x \in H$. Substituting $x = W_\circ$ we find $p_\circ(W_\circ) \equiv 13 W_\circ^3 \pmod{24}$; but since W_\circ is characteristic we have $W_\circ^3 \equiv 0 \pmod{2}$, thus $p_\circ(W_\circ) \equiv W_\circ^3 \pmod{24}$. Write $p_\circ(W_\circ) = W_\circ^3 + 24k$ for some $k \in \mathbf{Z}$.

case 1) $k \equiv 0 \pmod{2}$: define $p := p_\circ, T := 0$.

case 2) $k \equiv 1 \pmod{2}$: we must find a linear form $T_\circ \in H^\vee$ with $T_\circ(W_\circ) \equiv 1 \pmod{2}$; clearly this can be done if and only if W_\circ is not divisible by 2. If W_\circ were divisible by 2, $W_\circ = 2V_\circ$ for some $V_\circ \in H$, then $2p_\circ(V_\circ) = p_\circ(W_\circ) = W_\circ^3 + 24k = 8V_\circ^3 + 24k$ would give $p_\circ(V_\circ) = 4V_\circ^3 + 12k$; then, using $p_\circ(V_\circ) \equiv 4V_\circ^3 + 6V_\circ^2 W_\circ + 3V_\circ W_\circ^2 \equiv 4V_\circ^3 \pmod{24}$ we would find $k \equiv 0 \pmod{2}$, which is not the case by assumption.

This shows that $F \in S^3 H^\vee$ is realizable by a topological manifold with Pontrjagin class p_\circ and non-vanishing triangulation obstruction $\tau_\circ := \bar{T}_\circ \pmod{2}$. In order to realize F by a smooth manifold, one can take $p := p_\circ + 24T_\circ$, and $\tau := 0$.

REMARK 3. The topological counterpart of the existence of a characteristic element for a given cubic form $F \in S^3 H^\vee$ is the existence of a mod-2 Steenrod-algebra structure, which is a necessary condition for a ring to be a cohomology ring.

The existence and the classification of characteristic elements for a given cubic form is essentially a linear algebra problem over $\mathbf{Z}/_2$. To see this, let $F \in S^3 H^\vee$ be a fixed cubic form on a finitely generated free abelian group H . Associated with F we have a linear map $F^t : H \rightarrow S^2 H^\vee$ sending an element $h \in H$ to the bilinear form $F^t(h) : H \otimes H \rightarrow \mathbf{Z}, (x, y) \rightarrow x \cdot y \cdot h$. Let $\bar{H} := H/_2 H$. $\bar{F} \in S^3 \bar{H}^\vee$ be the reductions of H and F modulo 2, and let $- : H \rightarrow \bar{H}$ be the natural epimorphism. The symmetric trilinear form \bar{F} on the $\mathbf{Z}/_2$ -module \bar{H} defines a natural symmetric bilinear form $q_{\bar{F}} \in S^2 \bar{H}^\vee$ given by $q_{\bar{F}}(\bar{x}, \bar{y}) := \bar{x} \cdot \bar{y} \cdot (\bar{x} + \bar{y})$.

LEMMA 3. $F \in S^3 H^\vee$ admits characteristic elements if and only if $q_{\bar{F}}$ lies in the image of $\bar{F}^t \in \text{Hom}_{\mathbf{Z}}(H, S^2 \bar{H}^\vee)$. The set of all characteristic elements for F is a coset of the form $W_\circ + \text{Ker}(\bar{F}^t)$.

Proof. W_o is characteristic for F if and only if $q_{\bar{F}} = \bar{F}^t(W_o)$.

In terms of a \mathbf{Z} -basis $\{e_1, \dots, e_b\}$ for H the condition $q_{\bar{F}} \in \text{Im}(\bar{F}^t)$ translates into a simple rank condition over $\mathbf{Z}_{/2}$: the $\mathbf{Z}_{/2}$ -rank of the $b \times \binom{b+1}{2}$ -matrix A representing \bar{F}^t must be equal to the $\mathbf{Z}_{/2}$ -rank of the matrix A extended by the column $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leq i < j \leq b}$

EXAMPLE 3. Let $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ be free of rank 2, $F \in S^3 H^\vee$ given by $e_1^3 = a, e_1^2 e_2 = b, e_1 e_2^2 = c, e_2^3 = d$ with $a, b, c, d \in \mathbf{Z}$. The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \overline{b+c} \end{bmatrix}$$

2.2 HOMOTOPY TYPES WITH A GIVEN COHOMOLOGY RING

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr’s classification theorem we know that in algebraic terms this means the following: fix a non-negative integer r_o , a finitely generated free abelian group H_o , and a symmetric trilinear form $F_o \in S^3 H_o^\vee$ which admits characteristic elements.

Let $\mathcal{M}(r_o, H_o, F_o)$ be the set of 1-connected, closed, oriented, 6-dimensional manifolds X with $b_3(X) = 2r_o$, such that there exists an isomorphism $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$. Denote by $\text{Aut}(F_o)$ the subgroup of \mathbf{Z} -automorphisms of H_o which leave $F_o \in S^3 H_o^\vee$ invariant; $\text{Aut}(F_o)$ acts on pairs $(w, [l]) \in \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$ in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^* [l]) .$$

Let $\text{Aut}(F_o) \backslash \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$ be the set of $\text{Aut}(F_o)$ -orbits.

A manifold X in $\mathcal{M}(r_o, H_o, F_o)$ and an isomorphism $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$ yields a well-defined $\text{Aut}(F_o)$ -orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^* [p_1(X) + 24T]) \text{ (modulo } \text{Aut}(F_o) \text{) ,}$$

where $T \in H^4(X, \mathbf{Z})$ is an arbitrary integral lifting of $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$.

The set of oriented homotopy types $\mathcal{M}(r_o, H_o, F_o) / \simeq$ of manifolds in $\mathcal{M}(r_o, H_o, F_o)$ can now be described in the following way: