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Proof. W_o is characteristic for F if and only if $q_{\bar{F}} = \bar{F}^t(W_o)$.

In terms of a \mathbf{Z} -basis $\{e_1, \dots, e_b\}$ for H the condition $q_{\bar{F}} \in \text{Im}(\bar{F}^t)$ translates into a simple rank condition over $\mathbf{Z}_{/2}$: the $\mathbf{Z}_{/2}$ -rank of the $b \times \binom{b+1}{2}$ -matrix A representing \bar{F}^t must be equal to the $\mathbf{Z}_{/2}$ -rank of the matrix A extended by the column $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leq i < j \leq b}$

EXAMPLE 3. Let $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ be free of rank 2, $F \in S^3 H^\vee$ given by $e_1^3 = a, e_1^2 e_2 = b, e_1 e_2^2 = c, e_2^3 = d$ with $a, b, c, d \in \mathbf{Z}$. The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \overline{b+c} \end{bmatrix}$$

2.2 HOMOTOPY TYPES WITH A GIVEN COHOMOLOGY RING

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr’s classification theorem we know that in algebraic terms this means the following: fix a non-negative integer r_o , a finitely generated free abelian group H_o , and a symmetric trilinear form $F_o \in S^3 H_o^\vee$ which admits characteristic elements.

Let $\mathcal{M}(r_o, H_o, F_o)$ be the set of 1-connected, closed, oriented, 6-dimensional manifolds X with $b_3(X) = 2r_o$, such that there exists an isomorphism $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$. Denote by $\text{Aut}(F_o)$ the subgroup of \mathbf{Z} -automorphisms of H_o which leave $F_o \in S^3 H_o^\vee$ invariant; $\text{Aut}(F_o)$ acts on pairs $(w, [l]) \in \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$ in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^* [l]) .$$

Let $\text{Aut}(F_o) \backslash \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$ be the set of $\text{Aut}(F_o)$ -orbits.

A manifold X in $\mathcal{M}(r_o, H_o, F_o)$ and an isomorphism $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$ yields a well-defined $\text{Aut}(F_o)$ -orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^* [p_1(X) + 24T]) \text{ (modulo } \text{Aut}(F_o) \text{) ,}$$

where $T \in H^4(X, \mathbf{Z})$ is an arbitrary integral lifting of $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$.

The set of oriented homotopy types $\mathcal{M}(r_o, H_o, F_o) / \simeq$ of manifolds in $\mathcal{M}(r_o, H_o, F_o)$ can now be described in the following way:

PROPOSITION 3. *The assignment $X \mapsto (\alpha^{-1}(w_2(X)), \alpha^*[p_1(X) + 24T])$ (modulo $\text{Aut}(F_o)$) defines an injection.*

$$I: \mathcal{M}(r_o, H_o, F_o) / \cong \rightarrow_{\text{Aut}(F_o)} \backslash \bar{H}_o \times H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}.$$

Proof. Suppose X and X' are manifolds in $\mathcal{M}(r_o, H_o, F_o)$, $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ and $\alpha': H_o \rightarrow H^2(X', \mathbf{Z})$ isomorphisms with $\alpha^*F_X = F_o$ and $(\alpha')^*F_{X'} = F_o$. X and X' have the same image under I iff there exists an automorphism $\gamma \in \text{Aut}(F_o)$ with $\gamma\alpha^{-1}(w_2(X)) = (\alpha')^{-1}w_2(X')$ and $(\gamma^{-1})^*\alpha^*[p_1(X) + 24T] = (\alpha')^*[p_1(X') + 24T']$. Consider $\beta := \alpha \circ \gamma \circ \alpha^{-1}: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$; β is obviously an isomorphism with $\beta^*F_{X'} = F_X$, $\beta w_2(X) = w_2(X')$, and $\beta^*[p_1(X') + 24T'] = [p_1(X) + 24T]$; but this means that the systems of invariants associated with X and X' are weakly equivalent, and therefore X and X' oriented homotopy equivalent.

A complete description of the set $\mathcal{M}(r_o, H_o, F_o) / \cong$ i.e. of the image of I is only possible if the automorphism group $\text{Aut}(F_o)$ is known; this can be a serious problem, but we will see that the ‘general’ automorphism group is finite (and usually small), so that the next proposition gives a reasonable estimate for the number of elements in $\mathcal{M}(r_o, H_o, F_o) / \cong$.

PROPOSITION 4. *Fix $r_o \in \mathbf{N}$, a finitely generated free abelian group H_o , and a symmetric trilinear form $F_o \in S^3H_o^\vee$ which admits characteristic elements. Set $b := rk_{\mathbf{Z}}H_o$, $s := rk_{\mathbf{Z}/2}(\bar{F}_o^t)$, and let $t := rk_{\mathbf{Z}/2}(\cdot_{\bar{F}_o})$ be the $\mathbf{Z}/2$ -rank of the $\mathbf{Z}/2$ -linear square map $\cdot_{\bar{F}_o}: \bar{H}_o \rightarrow \bar{H}_o^\vee$ sending $\bar{u} \in \bar{H}_o$ to $\bar{u}^2 \in \bar{H}_o^\vee$. Then $\mathcal{M}(r_o, H_o, F_o) / \cong$ contains at most 2^{2b-s-t} elements.*

Proof. Fix any admissible system of invariants $(r_o, H_o, w_o, \tau_o, F_o, p_o)$ for a manifold in $\mathcal{M}(r_o, H_o, F_o)$. Given (r_o, H_o, F_o) , we know from the last lemma that the possible elements w_o form a coset of $\text{Ker}(\bar{F}_o^t)$ in \bar{H}_o , so that there exist precisely 2^{b-s} such elements. It remains to count the classes $[l] \in H_o^\vee /_{48H_o^\vee} /_{U_{F_o}}$, such that the $\text{Aut}(F_o)$ -orbit of $(w_o, [p_o + 24T_o + l])$ lies in the image of I .

To understand the latter condition we fix integral liftings $W_o, \in H_o, T_o \in H_o^\vee$ of w_o and τ_o satisfying the admissibility conditions

- i) $W_o^3 \equiv (p_o + 24T_o)(W_o) \pmod{48}$
- ii) $p_o(x) \equiv 4x^3 + 6x^2W_o + 3xW_o^2 \pmod{24} \quad \forall x \in H_o.$

Clearly the $\text{Aut}(F_o)$ -orbit of $(w_o, [p_o + 24T_o + l])$ lies in the image of I if and only if

i') $W_o^3 \equiv (p_o + 24T_o + l)(W_o) \pmod{48}$,

ii') $(p_o + l)(x) \equiv 4x^3 + 6x^2W_o + 3xW_o^2 \pmod{24} \quad \forall x \in H_o$,

which is equivalent to $l(W_o) \equiv 0 \pmod{48}$, and $l \equiv 0 \pmod{24H_o^\vee}$ because of i) and ii).

Now, by definition of the subgroup $U_{F_o} \subset H_o^\vee / 48H_o^\vee$ we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & \text{Ker}(\cdot \bar{F}_o) & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 \rightarrow \text{Ker}(24 \cdot \bar{F}_o) & \hookrightarrow & H_o / 2H_o & \xrightarrow{24 \cdot \bar{F}_o} & U_{F_o} & \rightarrow & 0 \\
 & & \cdot \bar{F}_o \downarrow & & \downarrow & & \\
 0 & \rightarrow & H_o^\vee / 2H_o^\vee & \xrightarrow{24} & H_o^\vee / 48H_o^\vee & \rightarrow & H_o^\vee / 24H_o^\vee \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & \text{Coker}(\cdot \bar{F}_o) & \rightarrow & H_o^\vee / 48H_o^\vee / U_{F_o} & \rightarrow & H_o^\vee / 24H_o^\vee \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The number of elements $[l] \in H_o^\vee / 48H_o^\vee / U_{F_o}$ to be counted coincides therefore with the cardinality of the kernel of the map $ev(w_o): \text{Coker}(\cdot \bar{F}_o) \rightarrow \mathbf{Z}_{/2}$ induced by evaluation in w_o . This number is at most $2^{b-t}(2^{b-t-1}$ if $w_o \neq 0$ and $t \neq b$).

COROLLARY 2. *If the $\mathbf{Z}_{/2}$ -rank $s = rk_{\mathbf{Z}_{/2}}(\cdot \bar{F}_o)$ is maximal, then $\mathcal{M}(r_o, H_o, F_o) / \cong$ contains at most one class.*

Proof. Suppose $\cdot \bar{F}_o: \bar{H}_o \rightarrow \bar{H}_o^\vee$ is surjective; then $\bar{F}_o^t: \bar{H}_o \rightarrow S^2 \bar{H}_o^\vee$ must have a trivial kernel, since $\bar{h}\bar{x}^2 = 0$ for all $\bar{x} \in \bar{H}_o$ implies $\bar{h} = 0$ if every linear form is a square. But this means $s = t = b$, so that $\mathcal{M}(r_o, H_o, F_o) / \cong$ has at most one element.

EXAMPLE 4. Let $H_o = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, $e_1^3 = a$, $e_1^2e_2 = b$, $e_1e_2^2 = c$, $e_2^3 = d$. If $\bar{b} \equiv \bar{c} \pmod{2}$, and $\bar{a}\bar{d} - \bar{b}\bar{c} \equiv 1 \pmod{2}$, then $\mathcal{M}(r_o, H_o, F_o) / \cong$ contains precisely one class for every $r_o \geq 0$.