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Proof. W_o is characteristic for F if and only if $q_{\bar{F}} = \bar{F}^t(W_o)$.

In terms of a $\mathbf{Z}$-basis $\{e_1, \ldots, e_b\}$ for H the condition $q_{\bar{F}} \in \mathrm{Im}(\bar{F}^t)$ translates into a simple rank condition over $\mathbf{Z}_{/2}$: the $\mathbf{Z}_{/2}$-rank of the $b \times \binom{b+1}{2}$-matrix A representing $\bar{F}^t$ must be equal to the $\mathbf{Z}_{/2}$-rank of the matrix A extended by the column $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leqslant i \leqslant j \leqslant b}$

EXAMPLE 3. Let $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ be free of rank 2, $F \in S^3 H^\vee$ given by $e_1^3 = a$, $e_1^2 e_2 = b$, $e_1 e_2^2 = c$, $e_2^3 = d$ with $a, b, c, d \in \mathbf{Z}$. The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \overline{b+c} \end{bmatrix}$$

2.2 Homotopy types with a given cohomology ring

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr's classification theorem we know that in algebraic terms this means the following: fix a non-negative integer r_o, a finitely generated free abelian group H_o, and a symmetric trilinear form $F_o \in S^3 H_o^\vee$ which admits characteristic elements.

Let $\mathcal{M}(r_o, H_o, F_o)$ be the set of 1-connected, closed, oriented, 6-dimensional manifolds X with $b_3(X) = 2r_o$, such that there exists an isomorphism $\alpha : H_o \to H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$. Denote by $\mathrm{Aut}(F_o)$ the subgroup of $\mathbf{Z}$-automorphisms of H_o which leave $F_o \in S^3 H_o^\vee$ invariant; $\mathrm{Aut}(F_o)$ acts on pairs $(w, [l]) \in \bar{H}_o \times H_o^\vee /_{48 H_o^\vee} /_{U_{F_o}}$ in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^* [l]) .$$

Let ${}_{\mathrm{Aut}(F_o)} \backslash \bar{H}_o \times H_o^\vee /_{48 H_o^\vee} /_{U_{F_o}}$ be the set of $\mathrm{Aut}(F_o)$-orbits.

A manifold X in $\mathcal{M}(r_o, H_o, F_o)$ and an isomorphism $\alpha : H_o \to H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$ yields a well-defined $\mathrm{Aut}(F_o)$-orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^* [p_1(X) + 24T]) \text{ (modulo } \mathrm{Aut}(F_o)) ,$$

where $T \in H^4(X, \mathbf{Z})$ is an arbitrary integral lifting of $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$.

The set of oriented homotopy types $\mathcal{M}(r_o, H_o, F_o)/_{\simeq}$ of manifolds in $\mathcal{M}(r_o, H_o, F_o)$ can now be described in the following way:

PROPOSITION 3. *The assignment* $X \mapsto (\alpha^{-1}(w_2(X)), \alpha^*[p_1(X)+24T])$ *(modulo* $\mathrm{Aut}(F_\circ)$*) defines an injection.*

$$I: \mathcal{M}(r_\circ, H_\circ, F_\circ)/_{\simeq} \to {}_{\mathrm{Aut}(F_\circ)}\backslash \bar{H}_\circ \times H^\vee_\circ/_{48H^\vee_\circ}/_{U_{F_\circ}}.$$

Proof. Suppose X and X' are manifolds in $\mathcal{M}(r_\circ, H_\circ, F_\circ)$, $\alpha: H_\circ \to H^2(X, \mathbf{Z})$ and $\alpha': H_\circ \to H^2(X', \mathbf{Z})$ isomorphisms with $\alpha^* F_X = F_\circ$ and $(\alpha')^* F_{X'} = F_\circ$. X and X' have the same image under I iff there exists an automorphism $\gamma \in \mathrm{Aut}(F_\circ)$ with $\gamma\alpha^{-1}(w_2(X)) = (\alpha')^{-1} w_2(X')$ and $(\gamma^{-1})^* \alpha^*[p_1(X)+24T] = (\alpha')^*[p_1(X')+24T']$. Consider $\beta := \alpha \circ \gamma \circ \alpha^{-1}: H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$; β is obviously an isomorphism with $\beta^* F_{X'} = F_X$, $\beta w_2(X) = w_2(X')$, and $\beta^*[p_1(X')+24T'] = [p_1(X)+24T]$; but this means that the systems of invariants associated with X and X' are weakly equivalent, and therefore X and X' oriented homotopy equivalent.

A complete description of the set $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$ i.e. of the image of I is only possible if the automorphism group $\mathrm{Aut}(F_\circ)$ is known; this can be a serious problem, but we will see that the 'general' automorphism group is finite (and usually small), so that the next proposition gives a reasonable estimate for the number of elements in $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$.

PROPOSITION 4. *Fix* $r_\circ \in \mathbf{N}$, *a finitely generated free abelian group* $H_\circ$, *and a symmetric trilinear form* $F_\circ \in S^3 H^\vee_\circ$ *which admits characteristic elements. Set* $b := rk_{\mathbf{Z}} H_\circ$, $s := rk_{\mathbf{Z}/2}(\bar{F}^t_\circ)$, *and let* $t := rk_{\mathbf{Z}/2}(\cdot_{\bar{F}_\circ})$ *be the* $\mathbf{Z}_{/2}$*-rank of the* $\mathbf{Z}_{/2}$*-linear square map* $\cdot_{\bar{F}_\circ}: \bar{H}_\circ \to \bar{H}^\vee_\circ$ *sending* $\bar{u} \in \bar{H}_\circ$ *to* $\bar{u}^2 \in \bar{H}^\vee_\circ$. *Then* $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$ *contains at most* 2^{2b-s-t} *elements.*

Proof. Fix any admissible system of invariants $(r_\circ, H_\circ, w_\circ, \tau_\circ, F_\circ, p_\circ)$ for a manifold in $\mathcal{M}(r_\circ, H_\circ, F_\circ)$. Given $(r_\circ, H_\circ, F_\circ)$, we know from the last lemma that the possible elements $w_\circ$ form a coset of $\mathrm{Ker}(\bar{F}^t_\circ)$ in $\bar{H}_\circ$, so that there exist precisely 2^{b-s} such elements. It remains to count the classes $[l] \in H^\vee_\circ/_{48H^\vee_\circ}/_{U_{F_\circ}}$, such that the $\mathrm{Aut}(F_\circ)$-orbit of $(w_\circ, [p_\circ + 24T_\circ + l])$ lies in the image of I.

To understand the latter condition we fix integral liftings $W_\circ, \in H_\circ$, $T_\circ \in H^\vee_\circ$ of $w_\circ$ and $\tau_\circ$ satisfying the admissibility conditions

i) $W^3_\circ \equiv (p_\circ + 24T_\circ)(W_\circ) \pmod{48}$

ii) $p_\circ(x) \equiv 4x^3 + 6x^2 W_\circ + 3xW^2_\circ \pmod{24}$ $\forall x \in H_\circ$.

Clearly the $\mathrm{Aut}(F_\circ)$-orbit of $(w_\circ, [p_\circ + 24T_\circ + l])$ lies in the image of I if and only if i′) $W_\circ^3 \equiv (p_\circ + 24T_\circ + l)\,(W_\circ)\ (\mathrm{mod}\ 48)$,

ii′) $(p_\circ + l)\,(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 (\mathrm{mod}\ 24)\ \ \forall x \in H_\circ$,

which is equivalent to $l(W_\circ) \equiv 0 (\mathrm{mod}\ 48)$, and $l \equiv 0 (\mathrm{mod}\ 24H_\circ^\vee)$ because of i) and ii).

Now, by definition of the subgroup $U_{F_\circ} \subset H_\circ^\vee/_{48H_\circ^\vee}$ we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & \mathrm{Ker}(\cdot_{\bar F_\circ}) & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \to \mathrm{Ker}(24\cdot_{\bar F_\circ}) & \hookrightarrow & H_\circ/_{2H_\circ} & \overset{24\cdot\bar F_\circ}{\to} & U_{F_\circ} & \to & 0 & & \\
 & & \cdot_{\bar F_\circ}\downarrow & & \downarrow & & & & \\
0 & \to & H_\circ^\vee/_{2H_\circ^\vee} & \overset{24}{\to} & H_\circ^\vee/_{48H_\circ^\vee} & \to & H_\circ^\vee/_{24H_\circ^\vee} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathrm{Coker}(\cdot_{\bar F_\circ}) & \to & H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}} & \to & H_\circ^\vee/_{24H_\circ^\vee} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The number of elements $[l] \in H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}}$ to be counted coincides therefore with the cardinality of the kernel of the map $ev(w_\circ)\colon \mathrm{Coker}(\cdot_{\bar F_\circ}) \to \mathbf{Z}_{/2}$ induced by evaluation in $w_\circ$. This number is at most $2^{b-t}(2^{b-t-1}$ if $w_\circ \neq 0$ and $t \neq b)$.

Corollary 2. *If the* $\mathbf{Z}_{/2}$*-rank* $s = rk_{\mathbf{Z}/2}(\cdot_{\bar F_\circ})$ *is maximal, then* $\mathscr{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$ *contains at most one class.*

Proof. Suppose $\cdot_{\bar F_\circ}\colon \bar H_\circ \to \bar H_\circ^\vee$ is surjective; then $\bar F_\circ^t\colon \bar H_\circ \to S^2 \bar H_\circ^\vee$ must have a trivial kernel, since $\bar h \bar x^2 = 0$ for all $\bar x \in \bar H_\circ$ implies $\bar h = 0$ if every linear form is a square. But this means $s = t = b$, so that $\mathscr{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$ has at most one element.

Example 4. Let $H_\circ = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, $e_1^3 = a$, $e_1^2 e_2 = b$, $e_1 e_2^2 = c$, $e_2^3 = d$. If $\bar b \equiv \bar c (\mathrm{mod}\ 2)$, and $\bar a \bar d - \bar b \bar c \equiv 1 (\mathrm{mod}\ 2)$, then $\mathscr{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$ contains precisely one class for every $r_\circ \geqslant 0$.