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The number of integral classes in these orbits is therefore finite. We have, however, an even stronger finiteness theorem for stable ternary cubics:

PROPOSITION 7. *Let H be a free $\mathbf{Z}$-module of rank 3. There exist only finitely many classes of symmetric trilinear forms $F \in S^3 H^\vee$ with a fixed discriminant $\Delta \neq 0$.*

Proof. In terms of Arnhold's invariants S and T, Δ is given by $\Delta = S^3 - T^2$. By a theorem of C. Siegel [Si], the diophantine equation $S^3 - T^2 = \Delta$ has only finitely many integral solution (S, T) for any integer $\Delta \neq 0$. For each of these solutions the corresponding point in $S^3 H_{\mathbf{C}}^\vee / _{SL(H_{\mathbf{C}})}$ lies outside of the discriminant curve, so that the π-fiber over it is a closed $SL(H_{\mathbf{C}})$-orbit. The finiteness of the class number then follows from the Borel/Harish-Chandra theorem.

A famous special case of Siegel's theorem is Bachet's equation $S^3 - T^2 = 2$; it has only the two obvious solutions $(3, \pm 5)$.

REMARK 10. To get finiteness results for ternary cubic forms it is not sufficient to fix the J-invariant (instead of the discriminant): The forms $f_m = X^3 + XZ^2 + Z^3 + mY^2Z$, $m \in \mathbf{Z}\backslash\{0\}$, all have the same J-invariant, but they are not equivalent, even over $\mathbf{Q}$, since they have bad reduction at different primes $p \mid m$.

4. INVARIANTS OF COMPLEX 3-FOLDS

In this section we begin to investigate the topology of 1-connected, compact, complex 3-folds. After a brief discussion of the possible systems of Chern numbers of almost complex 6-manifolds, we study the behaviour of the topological invariants of complex 3-folds under certain standard constructions, like e.g. branched coverings, or blow-ups of points and curves. Then we describe some interesting examples of 1-connected, non-Kählerian 3-folds, including a new construction method which generalizes the Calabi-Eckmann manifolds. These examples will be needed in the next section in order to realize complex types of cubic forms as cup-forms of complex 3-folds.

4.1 CHERN NUMBERS OF ALMOST COMPLEX STRUCTURES

Let X be a closed, oriented, 6-dimensional differentiable manifold. The tangent bundle of X is induced by a classifying map $t_X : X \to BSO(6)$ which is unique up to homotopy. By an almost complex structure on X we mean the homotopy class $[\tilde{t}_X]$ of a lifting $\tilde{t}_X : X \to BU(3)$ of t_X to $BU(3)$.

PROPOSITION 8. *Every closed, oriented, 6-dimensional C^∞-manifold X without 2-torsion in $H^3(X, \mathbf{Z})$ admits an almost complex structure. There is a 1-1 correspondence between almost complex structures on X and integral lifts $W \in H^2(X, \mathbf{Z})$ of $w_2(X)$. The Chern classes c_i of the almost complex manifold (X, W) are given by $c_1 = W$, $c_2 = \frac{1}{2}(W^2 - p_1(X))$.*

Proof (cf. [W]). The obstructions against lifting t_X to $BU(3)$ lie in the cohomology groups $H^{i+1}(X, \pi_i(SO(6)/_{U(3)}), i = 0, 1, \ldots, 5$. Since $SO(6)/_{U(3)} = \mathbf{P}^3$ has only one nontrivial homotopy group $\pi_2(SO(6)/_{U(3)}) \cong \mathbf{Z}$ in dimensions $i \leqslant 5$, there is in fact only one obstruction $o(t_X) \in H^3(X, \mathbf{Z})$, and this obstruction can be identified with the image of $w_2(X)$ under the Bockstein homomorphism $\beta : H^2(X, \mathbf{Z}_{/2}) \to H^3(X, \mathbf{Z})$. Since $H^3(X, \mathbf{Z})$ has no 2-torsion by assumption, $\beta w_2(X)$ must be equal to zero, so that X has at least one almost complex structure $[\tilde{t}_X] \in [X, BU(3)]$. Standard homotopy arguments show now that the map, which assigns to an almost complex structure $[\tilde{t}_X]$ its first Chern class $\tilde{t}_X^* c_1$, induces a 1-1 correspondence between integral lifts $W \in H^2(X, \mathbf{Z})$ of $w_2(X)$ and homotopy classes of liftings of $[t_X]$ to $BU(3)$.

The second Chern class c_2 of the almost complex manifold (X, W) is determined by $W^2 - 2c_2 = p_1(X)$.

The Chern numbers c_1^3, c_1c_2, c_3 of an almost complex manifold X of real dimension 6 satisfy the following congruences: $c_1^3 \equiv 0 (\mathrm{mod}\, 2)$, $c_1 c_2 \equiv 0 (\mathrm{mod}\, 24)$, $c_3 \equiv 0 (\mathrm{mod}\, 2)$. Conversely, given a triple (a, b, c) of integers $a \equiv 0 (\mathrm{mod}\, 2)$, $b \equiv 0 (\mathrm{mod}\, 24)$, and $c \equiv 0 (\mathrm{mod}\, 2)$, there always exists an almost complex manifold X of dimension 6 with Chern numbers $c_1^3 = a$, $c_1 c_2 = b$, $c_3 = c$.

It is not totally clear, however, that one can find a *connected* manifold X with prescribed Chern numbers [H1].

PROPOSITION 9. *Every triple $(a, b, c) \in \mathbf{Z}^{\oplus 3}$ satisfying $a \equiv 0 (\mathrm{mod}\, 2)$, $b \equiv 0 (\mathrm{mod}\, 24)$, $c \equiv 0 (\mathrm{mod}\, 2)$ is realizable as the Chern numbers of an almost complex 6-manifold.*

Proof. Consider the complete intersection $V(f, g) \subset \mathbf{P}^5$ defined by the polynomials $f(z) = z_0^2 + z_1^2 + 2z_2^2 - z_3^2 - z_4^2 - 2z_5^2$, and $g(z) = z_0^4 + z_1^4 + 2z_2^4 - z_3^4 - z_4^4 - 2z_5^4$ [We]. $V(f, g)$ is a singular 3-fold with 90 ordinary double points, and every small resolution V of these nodes is a (not necessarily projective) Calabi-Yau 3-fold with Euler number 4. Suppose now that a prescribed triple $(a, b, c) \in \mathbf{Z}^{\oplus 3}$ is realized by a possibly disconnected almost complex manifold $X = \coprod_{i \in I} X_i$. If we form the connected sum

X' of the X_i, we obtain a connected almost complex manifold X' with Chern numbers $c_1^3 = a$, $c_1c_2 = b$, but with $c_3 = c - 2(|I| - 1)$.

If $|I| > 1$ take the connected sum of X' with $|I| - 1$ copies of the complex manifold V. Since V is Calabi-Yau, the Chern numbers c_1^3 and c_1c_2 remain unchanged, whereas the Euler number of $X' \#_{|I|-1} V$ becomes $c_3 = c$.

REMARK 11. The above argument has been suggested by F. Hirzebruch after talk at the MPI, in which one of us had sketched a less geometric proof of the proposition.

There is another question which is related to the result above: Fix a closed, oriented, 6-dimensional differentiable manifold X. Which pairs (a, b) of integers with $a \equiv 0 \pmod 2$ and $b \equiv 0 \pmod{24}$ occur as Chern numbers c_1^3 and c_1c_2 of almost complex structures on X, and in how many ways?

For manifolds with $b_2(X) = 1$ the Chern numbers determine the almost complex structure. For manifolds with $b_2 > 1$ this is no longer true. It is possible to construct infinitely many distinct almost complex structures with the same Chern numbers on a hypersurface of bidegree (3, 3) in $\mathbf{P}^2 \times \mathbf{P}^2$.

An almost complex structure $[\tilde{t}_X]$ on a differentiable 6-manifold X is said to be integrable if $\tilde{t}_X$ is homotopic to the classifying map of a complex 3-fold. We are not aware of any example of an almost complex 6-manifold which is known not be integrable. On the other hand, it is also unknown whether or not the Chern numbers c_1^3, c_1c_2 of integrable almost complex manifold are topological invariants. The following remark might therefore be of some interest:

PROPOSITION 10. *If the Chern numbers of complex 3-folds are topological invariants, then there exist almost complex structures which are not integrable.*

Proof. Consider a closed, oriented differentiable 6-manifold X without 2-torsion in $H^3(X, \mathbf{Z})$. Fix any almost complex structure on X with first Chern class $W \in H^2(X, \mathbf{Z})$.

Every element $x \in H^2(X, \mathbf{Z})$ defines a new almost complex structure on X with first Chern class $W + 2x$, and it is easy to see that these two almost complex structures have the same Chern numbers if and only if x satisfies the equations $p_1(X) \cdot x = 0$, and $3W^2 \cdot x + 6W \cdot x^2 + 4x^3 = 0$.

Suppose now (X, W) is integrable, $p_1(X) \neq 0$, and choose $x \in H^2(X, \mathbf{Z})$ such that $p_1(X) \cdot x \neq 0$. Then clearly, either none of the almost complex manifolds $(X, W + 2x)$ is integrable, or the Chern numbers of complex 3-folds are not topologically invariant.

REMARK 12. It is very likely that there exist non-integrable almost complex structures on manifolds X as above, but probably this is hard to prove. It is also not unlikely that the Chern numbers of complex 3-folds are not topological invariants. A possible way to check this would be, to run a computer search for 3-folds given by certain standard constructions.

4.2 STANDARD CONSTRUCTIONS

For later use we investigate the topological invariants of complex 3-folds which can be obtained by certain simple standard constructions like complete intersections, simple cyclic coverings, blow-ups of points and curves, and projective bundles.

PROPOSITION 11 (Libgober/Wood). *Let* $X \subset \mathbf{P}^{3+r}$ *be a smooth complete intersection of multidegree* $\underline{d} = (d_1, \ldots, d_r)$. *Choose a normalized basis* $e \in H^2(X, \mathbf{Z})$, *and let* $\varepsilon \in H^4(X, \mathbf{Z})$ *be defined by* $\varepsilon(e) = 1$. *Then the invariants of* X *are:*

$F_X(xe) = dx^3$ *where* $d = \prod_{i=1}^r d_i$, $w_2(X) \equiv (4 + r - \sum_{i=1}^r d_i)e$,

$p_1(X) = d(4 + r - \sum_{i=1}^r d_i^2)\varepsilon$, *and*

$$b_3(X) = 4 - \tfrac{d}{6}[(4 + r - \textstyle\sum_{i=1}^r d_i)^3 - 3(4 + r - \sum_{i=1}^r d_i)(4 + r - \sum_{i=1}^r d_i^2) + 2(4 + r - \sum_{i=1}^r d_i^3)] .$$

Proof. [L/W].

PROPOSITION 12. *Let* X *be a smooth, 1-connected, complex projective 3-fold, and let* $\pi : X' \to X$ *be a simple cyclic covering of degree* d *branched along a non-singular ample divisor* $B \in |L^{\otimes d}|$. X' *is smooth, projective, 1-connected, and* $\pi^* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ *is an isomorphism. The invariants of* X *and* X' *are related by the formulae:*

$$(\pi^*)^* F_{X'} = dF_X,\ w_2(X') - \pi^* w_2(X) \equiv (d-1)\pi^* c_1(L),$$

$$p_1(X') - \pi^* p_1(X) = (1-d)(1+d)\pi^* c_1(L)^2, \text{ and}$$

$$b_3(X') = db_3(X) + (d-1)(b_2(B) - 2b_2(X)) .$$

Proof. X' is clearly smooth and projective. By a theorem of M. Cornalba $\pi : X' \to X$ is a 3-equivalence, i.e. $\pi_* : \pi_i(X') \to \pi_i(X)$ is bijective for $i \leqslant 2$, and surjective for $i = 3$ [Co]. X' is therefore 1-connected, and $\pi^* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ is an isomorphism. The relation between $F_{X'}$ and F_X is obvious, whereas the formula for $b_3(X')$ follows from $\pi_1(B) = \{1\}$ and standard properties of Euler numbers.