**Zeitschrift:** L'Enseignement Mathématique

**Herausgeber:** Commission Internationale de l'Enseignement Mathématique

**Band:** 41 (1995)

**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CUBIC FORMS AND COMPLEX 3-FOLDS

**Autor:** Okonek, Ch. / Van de Ven, A.

**Kapitel:** 5. Complex 3-folds with small $b_2$

**DOI:** https://doi.org/10.5169/seals-61829

EXAMPLE 13 (Twistor spaces). Let $p: Z \to M$ be the twistor fibration of a closed, oriented Riemannian 4-manifold $(M, g)$. $Z$ carries a natural almost complex structure which is integrable if and only if $g$ is self-dual [A/H/S]. Examples of 1-connected 4-manifolds which admit self-dual structures are $S^4, \#_n \mathbf{P}^2$, and $K3$-surfaces.

The total spaces of their twistor fibrations are 1-connected complex 3-folds which may be Moishezon for $S^4$ and $\#_n \mathbf{P}^2$ [C], but which are usually non-Kähler [Hi]. We leave it to the reader to calculate the topological invariants of these 3-folds. There is an interesting relation between Twistor spaces of connected sums and Kato's connection operation + for class $L$ manifolds [K2], [D/F].

EXAMPLE 14 (Oguiso). In a recent preprint [O1] K. Oguiso constructs examples of 1-connected, Moishezon Calabi-Yau 3-folds with very interesting cup-forms. He proves that for every integer $d \geqslant 1$ there exists a smooth complete intersection $X'_d$ of type (2, 4) in $\mathbf{P}^5$ which contains a non-singular rational curve $C_d$ of degree $d$ with normal bundle $N_{C_d/X_d} = \mathcal{O}_{C_d}(-1)^{\oplus 2}$.

The 3-fold $X'_d$ can now be flopped along $C_d$, i.e. $C_d$ can be blown up to $\mathbf{P}(N_{C_d/X_d}) \cong \mathbf{P}^1 \times \mathbf{P}^1$, and then 'blown down in the other direction'. The resulting 3-fold $X_d$ is a 1-connected Moishezon manifold with trivial canonical bundle and cup-form $F_{X_d}$ given by $F_{X_d}(xe_d) = (d^3 - 8)x^3$. Here $e_d \in H^2(X_d, \mathbf{Z})$ is the normalized generator corresponding to the strict transform of the negative of a hyperplane section of $X'_d$. The Pontrjagin class of $X_d$ is $p_1(X_d) = (112 + 4d)\varepsilon_d$ where $\varepsilon_d \in H^4(X_d, \mathbf{Z})$ denotes the generator with $\varepsilon_d(e_d) = 1$. Since the Euler-number does not change under a flop we have $b_3(X_d) = 180$ for every $d$.

## 5. COMPLEX 3-FOLDS WITH SMALL $b_2$

In this section we investigate the following natural problem: Which cubic forms can be realized as cup-forms of compact complex 3-folds? For small $b_2$ something can be said: Any core of a 1-connected, closed, oriented differentiable 6-manifold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ is homotopy equivalent to the core of a 1-connected complex 3-fold. In the case $b_2 = 2$, at least every discriminant $\Delta$ is realizable by a complex manifold. If $b_2 = 3$ we can realize all types of complex cubics with one exception, the union of a smooth conic and a tangent line. In addition to these realization results we prove a finiteness theorem for 3-folds with $b_2 = 1$, $w_2 \neq 0$, and we give examples which show that the condition $I_{F_X} \neq \emptyset$ for the index cone of a projective 3-fold with $h^{0,2} = 0$ is non-trivial in general.

## 5.1 3-FOLDS WITH $b_2 = 1$

Recall from section 1.1 that every closed, oriented, 1-connected differentiable 6-manifold $X$ with torsion-free homology has a connected sum decomposition $X \cong X_\circ \#_r S^3 \times S^3$ where $r = \left(\frac{b_3(X)}{2}\right)$, which is unique up to orientation preserving diffeomorphisms; the manifold $X_\circ$ with $b_3(X_\circ) = 0$ is the core of $X$.

THEOREM 6. *Let* $X_\circ$ *be a 1-connected, closed, oriented differentiable 6-manifold with* $H_2(X_\circ, \mathbf{Z}) \cong \mathbf{Z}$ *and* $b_3(X_\circ) = 0$. *There exists a compact complex 3-fold* $X$ *whose core is orientation preservingly homotopy equivalent to* $X_\circ$.

*Proof.* The oriented homotopy type of $X_\circ$ is determined by the invariants $d$, $w_2$, and $p_1 \pmod{48}$; more precisely: for $d \equiv 1 \pmod 2$ there is a single homotopy type whereas for $d \equiv 0 \pmod 2$ there are three; one of these 3 types has $w_2 \neq 0$, the other two are spin, they are distinguished by $p_1 \equiv 4d \pmod{48}$, $p_1 \equiv 4d + 24 \pmod{48}$ respectively. In order to realize these homotopy types as cores of complex 3-folds we first look at simple cyclic coverings of $\mathbf{P}^3$. Given a positive integer $d$, let $\pi : X \to \mathbf{P}^3$ be a simple cyclic covering of $\mathbf{P}^3$ branched along a smooth surface $B$ of degree $dl$. Then $X$ has the correct 'degree' $d$ and the characteristic classes $w_2 \equiv (d-1)l \pmod 2$, and $p_1 = 4d + (1-d)(1+d)dl^2$, see 4.2. For odd $d$ there is nothing to prove. For even $d$ we can realize $w_2 = 0$ or $w_2 \neq 0$ by choosing $l \equiv 0 \pmod 2$ or $l \equiv 1 \pmod 2$. Taking $l \equiv 0 \pmod 4$ gives $w_2 = 0$, $p_1 \equiv 4d \pmod{48}$, taking $l \equiv 2 \pmod 4$ yields $w_2 = 0$, and $p_1 \equiv 4d + 24 \pmod{48}$. It remains to treat the special case $d = 0$, where the 3 homotopy types are given by $w_2 \neq 0$, by $w_2 = 0$, $p_1 \equiv 0 \pmod{16}$, and by $w_2 = 0$, $p_1 \equiv 8 \pmod{16}$. The first two homotopy types are realizable as cores of elliptic fiber bundles over the projective plane blown up in two points.

The third homotopy type is realized by the core of Oguiso's Calabi-Yau 3-fold $X_2$ with vanishing cup-form and $p_1(X_2) = 120\varepsilon_2$.

The result just proven suggests a natural question: given a manifold $X_\circ$ as above, which (even) integers $b_3 \geqslant 0$ occur as the third Betti numbers of complex 3-folds $X$ whose core is homotopy equivalent to $X_\circ$?

There will certainly be some gaps for algebraic 3-folds. In order to show this, we prove the following finiteness theorem for families of Kähler structures:

THEOREM 7. *Fix a positive constant* $c$. *There exist only finitely many families of 1-connected, smooth projective 3-folds* $X$ *with* $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$, $w_2(X) \neq 0$, *and with* $b_3(X) \leqslant c$.

*Proof.* Let $X$ be a smooth projective 3-fold with $H_1(X, \mathbf{Z}) = \{0\}$, $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$, and with $w_2(X) \neq 0$. Clearly $\mathrm{Pic}(X) \cong H^2(X, \mathbf{Z})$, and we can choose a basis $e \in H^2(X, \mathbf{Z})$ corresponding to the ample generator of $\mathrm{Pic}(X)$.

Let $c_1(X) = c_1 e$, $c_2(X) = c_2 \varepsilon$ where $e^2 = d\varepsilon$, $\varepsilon(e) = 1$. If $c_1$ is positive, then $X$ is Fano, and there are only finitely many possibilities [Mu]. The case $c_1 = 0$ is excluded, so that we are left with $c_1 < 0$, i.e. the canonical bundle of $X$ is ample.

The Riemann-Roch formula $\chi(X, \mathcal{O}_X) = 1 - h^3(X, \mathcal{O}_X) = \frac{1}{24} c_1 c_2$ shows that the set of possible Chern numbers $c_1 c_2$ is bounded from below: $24(1 - c) \leqslant c_1 c_2$. Using Yau's inequality $8c_1(X)c_2(X) \leqslant 3c_1(X)^3$ we find that $d \mid c_1 \mid^3 \leqslant 64(c - 1)$, i.e. the degree $d$ and the order of divisibility $\mid c_1 \mid$ of $c_1(X)$ is bounded. Now Kollar's finiteness theorem [Ko2] yields the assertion.

EXAMPLE 15. Let $X$ be a 1-connected, smooth projective 3-fold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ and $w_2(X) \neq 0$. If $b_3(X) \leqslant 2$, then $h^3(X, \mathcal{O}_X) \leqslant 1$ and $X$ must be Fano of index 1 or 3. For $b_3(X) = 4$ we have that $X$ is either Fano, or $h^3(X, \mathcal{O}_X) = 2$ and $X$ is of general type with $d \mid c_1 \mid^3 \leqslant 64$.

Note that the assumption $w_2 \neq 0$ was only used to exclude Calabi-Yau 3-folds.

## 5.2 3-FOLDS WITH $b_2 = 2$

Let $X$ be a 1-connected, closed, oriented, 6-dimensional differentiable manifold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^2$.

We choose a basis $e_1, e_2$ for $H^2(X, \mathbf{Z})$ and set $a_0 = e_1^3$, $a_1 = e_1^2 e_2$, $a_2 = e_1 e_2^2$, $a_3 = e_2^3$; the cubic polynomial $f$ associated to the cup-form of $X$ is then given by $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 XY^2 + a_3 Y^3$. The discriminant of $f$ is by definition $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$; up to a factor it is simply the discriminant of the Hessian $H_f = 6^2[(a_0 a_2 - a_1^2)X^2 + (a_0 a_3 - a_1 a_2)XY + (a_1 a_3 - a_2^2)Y^2]$ of $f$: $\Delta(f) = (a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2)$.

The last identity shows that $\Delta(f)$ is always a square modulo 4, i.e. $\Delta(f) \equiv 0, 1 \pmod 4$.

PROPOSITION 17. *Every integer* $\Delta \equiv 0, 1 \pmod 4$ *is realizable as discriminant of a compact complex 3-fold.*

*Proof.* Consider the projectivization $X = \mathbf{P}_{\mathbf{P}^2}(E)$ of a holomorphic rank-2 vector bundle $E$ over the plane. In terms of the standard basis of $H^2(X, \mathbf{Z})$ $(e_1 = \pi^* h, e_2 = c_1(\mathcal{O}_{\mathbf{P}(E)}(1)))$ the cubic polynomial associated to $X$ is given by $f = (c_1^2 - c_2)X^3 + 3(-c_1)X^2 Y + 3XY^2$, where $c_i = c_i(E)$ are the Chern classes of $E$ considered as integers. Inserting this into the discriminant formula yields $\Delta(f) = c_1^2 - 4c_2$. Since every pair $c_1, c_2$ occurs as pair of Chern classes of a holomorphic rank-2 bundle on $\mathbf{P}^2$, every integer $\Delta \equiv 0, 1 \pmod 4$ can be realized as discriminant of a holomorphic projective bundle $\mathbf{P}_{\mathbf{P}^2}(E)$.

Recall from section 3.2 that there are 4 different types of $SL(2)$-orbits of complex binary cubics: non-singular forms $f$ (with $\Delta(f) \neq 0$), and three orbits of singular cubics, represented by the normal forms $X^2 Y, X^3$, and 0.

PROPOSITION 18. *All four types of complex binary cubics are realizable by complex 3-folds.*

*Proof.* We have seen this already for non-singular cubics. Clearly the product $\mathbf{P}^1 \times \mathbf{P}^2$ realizes the normal form $X^2 Y$. The cubics of normal forms $X^3$ or 0 are degenerate, i.e. their Hessians vanish identically. Therefore they can only be realized by non-Kählerian 3-folds. To realize $X^3$ one can blow up a point in an elliptic fiber bundle over a surface $Y$ with $b_2(Y) = 3$; the trivial form occurs for elliptic fiber bundles over a surface with $b_2 = 4$.

More detailed investigations of the possible homotopy types of real or complex manifolds with $b_2 = 2$ will appear elsewhere [Sch].

Here we only want to illustrate an interesting phenomenon which relates the ample cone of a projective 3-fold with $b_2 = 2$ to the Hessian of its cup-form.

PROPOSITION 19. *Let* $X$ *be a smooth projective 3-fold with* $b_2(X) = 2$. *The ample cone* $\mathscr{C}_X$ *is contained in the Hesse cone* $\mathscr{H}_F := \{h \in H^2(X, \mathbf{R}) \mid \det(F^t(h)) < 0\}$.

*Proof.* This is only a special case of our general result in section 4.3.

REMARK 14. The Hessian of a binary form $F \in S^3 H^\vee$ is identically zero iff $F$ is degenerate; it is negative semi-definite if $F$ is non-degenerate and $\Delta(F) \leqslant 0$; it is indefinite iff $\Delta(F) > 0$ [*Ca*]. Only in the indefinite case $\Delta(F) > 0$ can the closure $\bar{\mathscr{H}}_F := \{h \in H_{\mathbf{R}} \mid \det F'(h) \leqslant 0\}$ of the Hesse cone be a proper subset of $H_{\mathbf{R}}$.

EXAMPLE 16. Let $P = \mathbf{P}_{\mathbf{P}^2}(E)$ be the projectivization of a rank-2 vector bundle $E$ with Chern classes $c_i = c_i(E)$. The cup-form of $P$ yields the cubic polynomial $f = (c_1^2 - c_2)X^2 + 3(-c_1)X^2 Y + 3XY^2$ whose Hessian is $H_f = (-c_2)X^2 + c_1 XY - Y^2$. Rewriting $H_f$ as $H_f = -\frac{1}{4}[(2Y - c_1 X)^2 + X^2(4c_2 - c_1^2)] = \frac{-1}{4}[(2Y - c_1 X)^2 - \Delta(f)X^2]$ we find 3 possibilities for the Hesse cone:

i) $\Delta(f) < 0$: $\mathscr{H}_f = H^2(P, \mathbf{R}) \backslash \{0\}$

ii) $\Delta(f) = 0$: $\mathscr{H}_f = H^2(P, \mathbf{R}) \backslash L_{c_1}$ for a real line $L_{c_1}$ depending on $c_1$ ($L_{c_1} = \mathbf{R}(2, c_1)$ in the coordinates $X$, $Y$)

iii) $\Delta(f) > 0$: $\mathscr{H}_f$ is an open cone whose angle is determined by $\Delta(f)$ $((Z + \sqrt{\Delta(f)}X)(Z - \sqrt{\Delta(f)}X) > 0$ in coordinates $X, Z := 2Y - c_1 X)$.

## 5.3 3-FOLDS WITH $b_2 \geqslant 3$

Let $X$ be a 1-connected, compact complex 3-fold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 3}$. The cup-form of $X$ gives rise to a curve $C_X$ of degree 3 in the projective plane $\mathbf{P}(H^2(X, \mathbf{C}))$:

$$C_X := \{ \langle h \rangle \in \mathbf{P}(H^2(X, \mathbf{C})) \mid h^3 = 0 \}.$$

A first natural question is which types of plane cubic curves occur in this way?

Recall that there are 10 types of plane cubics, namely: 1) non-singular cubics, 2) irreducible cubics with a node, 3) irreducible cubics with a cusp, 4) reducible cubics consisting of a smooth conic and a transversal line, 5) smooth conics with a tangent line, 6) three lines forming a triangle, 7) three distinct lines through a common point, 8) a double line with a third skew line, 9) a triple line, 10) the trivial 'cubic' with equation 0.

LEMMA 4. *If the 3-fold $X$ has a non-trivial Hodge number $h^{2,0}(X) \neq 0$, then $C_X$ is of type 4), 6) 9) or 10).*

*Proof.* Choose basis vectors $e^{k,l} \in H^{k,l}(X)$, so that every $h \in H^2(X, \mathbf{C})$ can be uniquely written as $h = xe^{2,0} + ye^{1,1} + ze^{0,2}$.

Then clearly $h^3 = y[y^2(e^{1,1})^3 + 6xz(e^{2,0} \cdot e^{1,1} \cdot e^{0,2})]$.

We now realize the cubics of types 7)-10). These cubics are degenerate, i.e. they are cones, and therefore their Hessians vanish identically. From section 4.3 we know that they can not be realized by Kählerian 3-folds.

PROPOSITION 20. *The plane cubics of types 7)-10) can all be realized by 1-connected, non-Kählerian 3-folds.*

*Proof.* 'Cubics' of type 10) can be realized by elliptic fibre bundles over surfaces $Y$ with $b_2(Y) = 5$. In order to realize cubics of type 9) or 7) one blows up one or two points in an elliptic fibre bundle over a surface with $b_2 = 4$ or 3 respectively. The realization of a type 8) cubic is a little trickier: One starts with an elliptic fibre bundle over a surface $Y$ with $b_2(Y) = 3$, and blows up one of its fibers. The resulting 3-fold $X'$ has $b_2(X') = 2$ and $F_{X'} \equiv 0$. Now choose a line $l$ in the exceptional divisor $E$ of $X'$, and let $X$ be the blow-up of $X'$ along $l$. The cup-form of $X$ yields the cubic polynomial $x^2[y(-3l \cdot E) - x(\deg N_{C/X'})]$ with a non-zero coefficient $-3l \cdot E = 3$.

There are four types of complex cubics which we have been able to realize by projective 3-folds.

PROPOSITION 21. *Cubics of type 1), 3), 4) and 6) are realizable by 1-connected projective 3-folds.*

*Proof.* Type 1) occurs for blow-ups of complete intersections in two distinct points. The product $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ realizes a triangle, whereas most projective bundles over a surface with $b_2 = 2$ lead to the union of a smooth conic and a transversal line.

Irreducible cubics with a cusp can be obtained by blowing-up a line and a point in $\mathbf{P}^3$. The resulting 3-fold yields the cubic polynomial $X^3 - 3XY^2 - 2Y^3 + Z^3 = (X + Y)^2(X - 2Y) + Z^3$.

The remaining two types of cubics are cubics with a node (type 2)), and smooth conics with a tangent line (type 5)). We do not know if these types are realizable by projective 3-folds. A non-Kählerian 3-fold whose cup-form yields a nodal cubic can be constructed: one just takes the blow-up of two suitable curves in Oguiso's Calabi-Yau 3-fold with $b_2 = 1$ and vanishing cup-form.

Finally we like to show that the non-emptiness condition on the index cone of a projective 3-fold with $h^{0,2} = 0$ gives non-trivial restrictions for the possible cup-forms if $b_2 \geqslant 4$. Further investigations of this condition will appear elsewhere [Sch].

EXAMPLE 17. Let $H$ be a free $\mathbf{Z}$-module of rank 4 with basis $(e_i)_{i=1,\ldots,4}$. Consider a trilinear form $F \in S^3 H^\vee$ and its adjoint map $F^t : H \to S^2 H^\vee$. The image $F^t(h)$ of an element $h \in H$ is in terms of the chosen basis $(e_i)_{i=1,\ldots,4}$ represented by the symmetric $4 \times 4$-matrix $[[he_i e_j]]_{i,j=1,\ldots,4}$. Suppose this matrix is a diagonal sum $[[he_i e_j]]_{i,j=1,2} \oplus [[he_k e_l]]_{k,l=3,4}$ such that the determinants of both $2 \times 2$-matrices are negative for every $h \in H \backslash \{0\}$.

In this case $F^t(h)$ were of signature $(1, -1, 1, -1)$ for every $h \in H \backslash \{0\}$, and we would have $I_F = \mathscr{H}_F = \emptyset$.

All these conditions can be met, e.g. by setting $e_1^2 e_2 = e_2^3 = e_3^2 e_4 = e_4^3 = 1$, $e_1 e_2^2 = e_3 e_4^2 = 2$, and $e_i e_j e_k = 0$ otherwise. In this particular case the image of $h = \sum_{i=1}^4 h_i e_i$ under $F^t$ is represented by the matrix

$$\left[\begin{array}{cc|cc} h_2 & h_1 + 2h_2 & 0 & 0 \\ h_1 + 2h_2 & 2h_1 + h_2 & 0 & 0 \\ \hline 0 & 0 & h_4 & h_3 + 2h_4 \\ 0 & 0 & h_3 + 2h_4 & 2h_3 + h_4 \end{array}\right],$$

which has a positive determinant unless $h = 0$.

## REFERENCES


[A] ARNDT, F. Zur Theorie der binären kubischen Formen. *Crelle's Journal 53,* 309-321 (1857).

[Ar1] ARONHOLD, S. Zur Theorie der homogenen Funktionen dritten Grades von drei Variabeln. *Crelle's Journal 39,* 140-159 (1850).

[Ar2] —— Theorie der homogenen Funktionen dritten Grades von drei Veränderlichen. *Crelle's Journal 55,* 93-191 (1858).

[At] ATIYAH, M. Some examples of complex manifolds. *Bonner Math. Schriften 5,* 1-28 (1958).

[A/H/S] ATIYAH, M., N. HITCHIN and I. SINGER. Self-duality in four-dimensional Riemannian geometry. *Proc. R. Soc. Lond. A. 362,* 425-461 (1978).