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There will certainly be some gaps for algebraic 3-folds. In order to show this, we prove the following finiteness theorem for families of Kähler structures:

THEOREM 7. *Fix a positive constant* c. *There exist only finitely many families of 1-connected, smooth projective 3-folds* X *with* $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$, $w_2(X) \neq 0$, *and with* $b_3(X) \leqslant c$.

Proof. Let X be a smooth projective 3-fold with $H_1(X, \mathbf{Z}) = \{0\}$, $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$, and with $w_2(X) \neq 0$. Clearly $\operatorname{Pic}(X) \cong H^2(X, \mathbf{Z})$, and we can choose a basis $e \in H^2(X, \mathbf{Z})$ corresponding to the ample generator of $\operatorname{Pic}(X)$.

Let $c_1(X) = c_1 e$, $c_2(X) = c_2 \varepsilon$ where $e^2 = d\varepsilon$, $\varepsilon(e) = 1$. If c_1 is positive, then X is Fano, and there are only finitely many possibilities [Mu]. The case $c_1 = 0$ is excluded, so that we are left with $c_1 < 0$, i.e. the canonical bundle of X is ample.

The Riemann-Roch formula $\chi(X, \mathcal{O}_X) = 1 - h^3(X, \mathcal{O}_X) = \frac{1}{24} c_1 c_2$ shows that the set of possible Chern numbers $c_1 c_2$ is bounded from below: $24(1 - c) \leqslant c_1 c_2$. Using Yau's inequality $8c_1(X) c_2(X) \leqslant 3c_1(X)^3$ we find that $d \,|\, c_1 \,|^3 \leqslant 64(c - 1)$, i.e. the degree d and the order of divisibility $|\, c_1 \,|$ of $c_1(X)$ is bounded. Now Kollar's finiteness theorem [Ko2] yields the assertion.

EXAMPLE 15. Let X be a 1-connected, smooth projective 3-fold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ and $w_2(X) \neq 0$. If $b_3(X) \leqslant 2$, then $h^3(X, \mathcal{O}_X) \leqslant 1$ and X must be Fano of index 1 or 3. For $b_3(X) = 4$ we have that X is either Fano, or $h^3(X, \mathcal{O}_X) = 2$ and X is of general type with $d \,|\, c_1 \,|^3 \leqslant 64$.

Note that the assumption $w_2 \neq 0$ was only used to exclude Calabi-Yau 3-folds.

5.2 3-FOLDS WITH $b_2 = 2$

Let X be a 1-connected, closed, oriented, 6-dimensional differentiable manifold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^2$.

We choose a basis e_1, e_2 for $H^2(X, \mathbf{Z})$ and set $a_0 = e_1^3$, $a_1 = e_1^2 e_2$, $a_2 = e_1 e_2^2$, $a_3 = e_2^3$; the cubic polynomial f associated to the cup-form of X is then given by $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 XY^2 + a_3 Y^3$. The discriminant of f is by definition $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$; up to a factor it is simply the discriminant of the Hessian $H_f = 6^2[(a_0 a_2 - a_1^2)X^2 + (a_0 a_3 - a_1 a_2)XY + (a_1 a_3 - a_2^2)Y^2]$ of f: $\Delta(f) = (a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2)$.

The last identity shows that $\Delta(f)$ is always a square modulo 4, i.e. $\Delta(f) \equiv 0, 1 \pmod 4$.

PROPOSITION 17. *Every integer* $\Delta \equiv 0, 1 \pmod 4$ *is realizable as discriminant of a compact complex 3-fold.*

Proof. Consider the projectivization $X = \mathbf{P}_{\mathbf{P}^2}(E)$ of a holomorphic rank-2 vector bundle E over the plane. In terms of the standard basis of $H^2(X, \mathbf{Z})$ $(e_1 = \pi^* h, e_2 = c_1(\mathcal{O}_{\mathbf{P}(E)}(1)))$ the cubic polynomial associated to X is given by $f = (c_1^2 - c_2)X^3 + 3(-c_1)X^2Y + 3XY^2$, where $c_i = c_i(E)$ are the Chern classes of E considered as integers. Inserting this into the discriminant formula yields $\Delta(f) = c_1^2 - 4c_2$. Since every pair c_1, c_2 occurs as pair of Chern classes of a holomorphic rank-2 bundle on $\mathbf{P}^2$, every integer $\Delta \equiv 0, 1 \pmod 4$ can be realized as discriminant of a holomorphic projective bundle $\mathbf{P}_{\mathbf{P}^2}(E)$.

Recall from section 3.2 that there are 4 different types of $SL(2)$-orbits of complex binary cubics: non-singular forms f (with $\Delta(f) \neq 0$), and three orbits of singular cubics, represented by the normal forms X^2Y, X^3, and 0.

PROPOSITION 18. *All four types of complex binary cubics are realizable by complex 3-folds.*

Proof. We have seen this already for non-singular cubics. Clearly the product $\mathbf{P}^1 \times \mathbf{P}^2$ realizes the normal form X^2Y. The cubics of normal forms X^3 or 0 are degenerate, i.e. their Hessians vanish identically. Therefore they can only be realized by non-Kählerian 3-folds. To realize X^3 one can blow up a point in an elliptic fiber bundle over a surface Y with $b_2(Y) = 3$; the trivial form occurs for elliptic fiber bundles over a surface with $b_2 = 4$.

More detailed investigations of the possible homotopy types of real or complex manifolds with $b_2 = 2$ will appear elsewhere [Sch].

Here we only want to illustrate an interesting phenomenon which relates the ample cone of a projective 3-fold with $b_2 = 2$ to the Hessian of its cup-form.

PROPOSITION 19. *Let* X *be a smooth projective 3-fold with* $b_2(X) = 2$. *The ample cone* $\mathcal{C}_X$ *is contained in the Hesse cone* $\mathcal{H}_F := \{h \in H^2(X, \mathbf{R}) \mid \det(F^t(h)) < 0\}$.

Proof. This is only a special case of our general result in section 4.3.

REMARK 14. The Hessian of a binary form $F \in S^3 H^\vee$ is identically zero iff F is degenerate; it is negative semi-definite if F is non-degenerate and $\Delta(F) \leqslant 0$; it is indefinite iff $\Delta(F) > 0$ [*Ca*]. Only in the indefinite case $\Delta(F) > 0$ can the closure $\overline{\mathscr{H}}_F := \{h \in H_{\mathbf{R}} \mid \det F^t(h) \leqslant 0\}$ of the Hesse cone be a proper subset of $H_{\mathbf{R}}$.

EXAMPLE 16. Let $P = \mathbf{P}_{\mathbf{P}^2}(E)$ be the projectivization of a rank-2 vector bundle E with Chern classes $c_i = c_i(E)$. The cup-form of P yields the cubic polynomial $f = (c_1^2 - c_2)X^2 + 3(-c_1)X^2Y + 3XY^2$ whose Hessian is $H_f = (-c_2)X^2 + c_1XY - Y^2$. Rewriting H_f as $H_f = -\frac{1}{4}[(2Y - c_1X)^2 + X^2(4c_2 - c_1^2)] = \frac{-1}{4}[(2Y - c_1X)^2 - \Delta(f)X^2]$ we find 3 possibilities for the Hesse cone:

i) $\Delta(f) < 0$: $\mathscr{H}_f = H^2(P, \mathbf{R}) \backslash \{0\}$

ii) $\Delta(f) = 0$: $\mathscr{H}_f = H^2(P, \mathbf{R}) \backslash L_{c_1}$ for a real line L_{c_1} depending on c_1 ($L_{c_1} = \mathbf{R}(2, c_1)$ in the coordinates X, Y)

iii) $\Delta(f) > 0$: $\mathscr{H}_f$ is an open cone whose angle is determined by $\Delta(f)$ ($(Z + \sqrt{\Delta(f)}X)(Z - \sqrt{\Delta(f)}X) > 0$ in coordinates $X, Z := 2Y - c_1X$).

5.3 3-FOLDS WITH $b_2 \geqslant 3$

Let X be a 1-connected, compact complex 3-fold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 3}$. The cup-form of X gives rise to a curve C_X of degree 3 in the projective plane $\mathbf{P}(H^2(X, \mathbf{C}))$:

$$C_X := \{\langle h \rangle \in \mathbf{P}(H^2(X, \mathbf{C})) \mid h^3 = 0\}.$$

A first natural question is which types of plane cubic curves occur in this way?

Recall that there are 10 types of plane cubics, namely: 1) non-singular cubics, 2) irreducible cubics with a node, 3) irreducible cubics with a cusp, 4) reducible cubics consisting of a smooth conic and a transversal line, 5) smooth conics with a tangent line, 6) three lines forming a triangle, 7) three distinct lines through a common point, 8) a double line with a third skew line, 9) a triple line, 10) the trivial 'cubic' with equation 0.

LEMMA 4. *If the 3-fold* X *has a non-trivial Hodge number* $h^{2,0}(X) \neq 0$, *then* C_X *is of type 4), 6) 9) or 10).*