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Autor: Geoghegan, Ross / Nicas, Andrew

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because G' is normal in G. In particular, $\varepsilon_*(\mathrm{trans}(C(g))) = n$ if $g \in G'$. There is a commutative diagram:

$$\begin{array}{ccc} K_0(\mathbf{Q}G) & \overset{T_0}{\to} & HH_0(\mathbf{Q}G) \\ \mathrm{res}\downarrow & & \mathrm{trans}\downarrow \\ K_0(\mathbf{Q}G') & \overset{T_0}{\to} & HH_0(\mathbf{Q}G') \end{array}$$

where $\mathrm{res}: K_0(\mathbf{Q}G) \to K_0(\mathbf{Q}G')$ is obtained by regarding a projective $\mathbf{Q}G$ module as a projective $\mathbf{Q}G'$ module; see [Bass] for details concerning the finite index transfer.

Recall that $HH_0(\mathbf{Q}G) = HH_0(\mathbf{Q}G)_H \oplus HH_0(\mathbf{Q}G)'_H$ where $HH_0(\mathbf{Q}G)'_H$ is the direct sum of the summands $HH_0(\mathbf{Q}G)_{C(g)}$ corresponding to the conjugacy classes not represented by elements of H. By Schafer's theorem [Sch, p. 224] applied to the normal subgroup $H \subset G$, the image of $T_0: K_0(\mathbf{Q}G) \to HH_0(\mathbf{Q}G)$ lies in $HH_0(\mathbf{Q}G)_H$. Thus we can replace $HH_0(\mathbf{Q}G)$ with $HH_0(\mathbf{Q}G)_H$ in the above diagram and obtain the commutative diagram:

$$\begin{array}{ccccc} K_0(\mathbf{Q}G) & \overset{T_0}{\to} & HH_0(\mathbf{Q}G)_H & \overset{\varepsilon_*}{\to} & \mathbf{Q} \\ \mathrm{res}\downarrow & & \mathrm{trans}\downarrow & & \times n\downarrow \\ K_0(\mathbf{Q}G') & \overset{T_0}{\to} & HH_0(\mathbf{Q}G') & \overset{\varepsilon_*}{\to} & \mathbf{Q} \end{array}$$

(the right square commutes because $H \subset G'$ and because of the observation made above). Write $HH_0(\mathbf{Q}G)_H = HH_0(\mathbf{Q}G)_{C(1)} \oplus HH_0(\mathbf{Q}G)''_H$ where $HH_0(\mathbf{Q}G)''_H$ is the direct sum of the $HH_0(\mathbf{Q}G)_{C(g)}$'s over $C(g) \in c(H) - \{C(1)\}$; also, $HH_0(\mathbf{Q}G') = HH_0(\mathbf{Q}G')_{C(1)} \oplus HH_0(\mathbf{Q}G')'$. Then $\mathrm{trans}(HH_0(\mathbf{Q}G)_{C(1)}) \subset HH_0(\mathbf{Q}G')_{C(1)}$ and $\mathrm{trans}(HH_0(\mathbf{Q}G)''_H) \subset HH_0(\mathbf{Q}G')'$. By Lemma 8.6, G' has the WBP over $\mathbf{Q}$, i.e. the composite $K_0(\mathbf{Q}G') \overset{T_0}{\to} HH_0(\mathbf{Q}G') \to HH_0(\mathbf{Q}G')' \overset{\varepsilon_*}{\to} \mathbf{Q}$ is zero. The conclusion follows from the above diagram. □

9. Trace formulae for homological intersections

The goal of this section is to prove a "trace formula" (Theorem 9.13) for the homological intersection of the graph of a map $F: M \times Y \to M$ with the graph of the projection map $p: M \times Y \to M$ where Y is a closed oriented manifold and M is a compact oriented manifold. This result will be applied in § 10 to complete the proof of Theorem 1.1.

In what follows, all homology and cohomology groups will have coefficients in a field $\mathbf{F}$. Recall that we use Dold's sign conventions [D2] for cup, cap and cross products.

Let M be a compact n-dimensional manifold with boundary ∂M. Assume M is oriented over $\mathbf{F}$ with fundamental class $[M] \in H_n(M, \partial M)$. Let $V \subset M$ be an open collar of ∂M and $M' = M - V$. Let $\Delta \subset M \times M$ be the diagonal and let

$$(M \times M, M' \times \partial M) \overset{j}{\hookrightarrow} (M \times M, M \times M - \Delta) \quad \text{and}$$

$$(M \times M, M' \times \partial M) \overset{i}{\hookrightarrow} (M \times M, M \times \partial M)$$

be the inclusions. Since i is a homotopy equivalence of pairs it induces an isomorphism $i^*: H^*(M \times M, M \times \partial M) \to H^*(M \times M, M' \times \partial M)$. We define the *diagonal cohomology class* $D_M \in H^n(M \times M, M \times \partial M)$ by $D_M \equiv (i^*)^{-1} j^*(T_M)$ where

$$T_M \in H^n(M \times M, M \times M - \Delta)$$

is the Thom class of M (see [Sp, §6.2] where T_M is called an *orientation* for M).

There is a slant product $H^i(M \times M, M \times \partial M) \otimes H_j(M, \partial M) \xrightarrow{/} H^{i-j}(M)$, see [MS, p. 125]. The reader should be aware that the sign conventions for cup, cap and cross products used in [MS] coincide with those of [D2] but differ from those of [Sp]. A straightforward adaptation of the proof of [MS, Lemma 11.9], where the case $\partial M = \emptyset$ is treated, shows that the fundamental class of M and the diagonal cohomology class of M are related by:

PROPOSITION 9.1. $D_M/[M] = 1 \in H^0(M)$. □

For each $k \geqslant 0$, choose a basis $\{b_j^k \mid j = 1, \dots, N(k)\}$ for $H_k(M)$. Let $\{\bar{b}_j^k \mid j = 1, \dots, N(k)\}$, be the corresponding dual basis for $H^k(M)$, i.e. $\langle \bar{b}_i^k, b_j^k \rangle = \delta_{ij}$ (Kronecker delta). For $k \geqslant 0$, define $d_j^{n-k} \in H^{n-k}(M, \partial M)$, $j = 1, \dots, N(k)$, by $b_j^k = d_j^{n-k} \cap [M]$. The proof of [MS, Theorem 11.11] carries over directly to show:

PROPOSITION 9.2. $D_M = \sum_{k \geqslant 0} (-1)^k \sum_{i=1}^{N(k)} \bar{b}_i^k \times d_i^{n-k}$. □

Let Y be a parameter space (Y is not required to be a manifold). Let $F: M \times Y \to M$ be a map. For $\alpha \in H_q(Y)$, define $f_{ij}^k(\alpha) \in \mathbf{F}$ by $F_*(b_j^k \times \alpha) = \sum_{i=1}^{N(k+q)} f_{ij}^k(\alpha) b_i^{k+q}$.

The Künneth Theorem allows us to write

$$F^*(\bar{b}_j^k) = \sum_{s=0}^{k} \sum_{l=1}^{N(s)} \bar{b}_l^s \times \omega(k,j,s,l)$$

where $\omega(k,j,s,l) \in H^{k-s}(Y)$.

LEMMA 9.3. $f_{ij}^k(\alpha) = (-1)^{qk}\langle \omega(k+q,i,k,j), \alpha \rangle$.

Proof. We have:

$$\begin{aligned} f_{ij}^k(\alpha) &= \langle \bar{b}_i^{k+q}, F_*(b_j^k \times \alpha)\rangle = \langle F^*(\bar{b}_i^{k+q}), b_j^k \times \alpha \rangle \\ &= \sum_{s=0}^{k+q} \sum_{l=1}^{N(s)} \langle \bar{b}_l^s \times \omega(k+q,i,s,l), b_j^k \times \alpha \rangle \\ &= \sum_{s=0}^{k+q} \sum_{l=1}^{N(s)} (-1)^{(k+q-s)k} \langle \bar{b}_l^s \cap b_j^k, \omega(k+q,i,s,l) \cap \alpha \rangle \\ &= (-1)^{qk}\langle \omega(k+q,i,k,j), \alpha \rangle . \qquad \square \end{aligned}$$

Let $\bar{F}: M \times Y \to M \times M$ be defined by $\bar{F}(m,y) = (F(m,y), m)$ and let $p: M \times Y \to M$ be projection. We define the *intersection invariant of F* to be the degree 0 homomorphism $\bar{I}(F): H_*(Y) \to H_*(M)$ given by $\bar{I}(F)(\alpha) = p_*(\bar{F}^*(D_M) \cap ([M] \times \alpha)) \in H_q(M)$ where $\alpha \in H_q(Y)$.

PROPOSITION 9.4. *For any* $\alpha \in H_q(Y)$,

$$\bar{I}(F)(\alpha) = \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \bar{b}_j^k \cap F_*(b_j^k \times \alpha) .$$

Proof. We have:

$$\begin{aligned} \bar{F}^*(\bar{b}_j^k \times d_j^{n-k}) &= F^*(\bar{b}_j^k) \cup (d_j^{n-k} \times 1) \\ &= \left(\sum_{s=0}^{k} \sum_{l=1}^{N(s)} \bar{b}_l^s \times \omega(k,j,s,l) \right) \cup (d_j^{n-k} \times 1) \\ &= \sum_{s=0}^{k} \sum_{l=1}^{N(s)} (-1)^{(k-s)(n-k)} (\bar{b}_l^s \cup d_j^{n-k}) \times \omega(k,j,s,l) . \end{aligned}$$

Now $(\bar{b}_l^s \cup d_j^{n-k}) \cap [M] = \bar{b}_l^s \cap (d_j^{n-k} \cap [M]) = \bar{b}_l^s \cap b_j^k$ and thus

$$\bar{F}^*(\bar{b}_j^k \times d_j^{n-k}) \cap ([M] \times \alpha)$$

$$= \sum_{s=0}^{k} \sum_{l=1}^{N(s)} (-1)^{(k-s)(n-k)} (-1)^{(k-s)n} ((\bar{b}_l^s \cup d_j^{n-k}) \cap [M]) \times (\omega(k,j,s,l) \cap \alpha)$$

$$= \sum_{s=0}^{k} \sum_{l=1}^{N(s)} (-1)^{(k-s)k} (\bar{b}_l^s \cap b_j^k) \times (\omega(k,j,s,l) \cap \alpha) .$$

Using Proposition 9.2 and the above identity, we obtain:

$$\bar{F}^*(D_M) \cap ([M] \times \alpha)$$

$$= \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \sum_{s=0}^{k} \sum_{l=1}^{N(s)} (-1)^{(k-s)k} (\bar{b}_l^s \cap b_j^k) \times (\omega(k,j,s,l) \cap \alpha) .$$

Now $p_*((\omega(k,j,s,l) \cap \alpha) \times (\bar{b}_l^s \cap b_j^k)) = 0$ unless $k - s = q$. Thus

$$p_*(\bar{F}^*(D_M) \cap ([M] \times \alpha))$$

$$= \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \sum_{l=1}^{N(k-q)} (-1)^{qk} \langle \omega(k,j,k-q,l), \alpha \rangle (\bar{b}_l^{k-q} \cap b_j^k) .$$

Since $\bar{b}_l^{k-q} = 0$ for $k - q < 0$, we can rewrite the above expression using the index variable $r = k - q$ as:

$$p_*(\bar{F}^*(D_M) \cap ([M] \times \alpha))$$

(9.5)

$$= \sum_{r \geqslant 0} (-1)^r \sum_{j=1}^{N(r+q)} \sum_{l=1}^{N(r)} (-1)^{qr} \langle \omega(r+q,j,r,l), \alpha \rangle (\bar{b}_l^r \cap b_j^{r+q}) .$$

Using Lemma 9.3,

$$\sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \bar{b}_j^k \cap F_*(b_j^k \times \alpha) = \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \sum_{i=1}^{N(k+q)} f_{ij}^k(\alpha)(\bar{b}_j^k \cap b_i^{k+q})$$

$$= \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \sum_{i=1}^{N(k+q)} (-1)^{qk} \langle \omega(k+q,i,k,j), \alpha \rangle (\bar{b}_j^k \cap b_i^{k+q}) .$$

Clearly, this last expression is the same as (9.5). □

We define the *diagonal homology class* $\Delta_M \in H_n(M \times M, \partial M \times M)$ by $\Delta_M \equiv \Delta_*([M])$ where Δ is the diagonal map $\Delta(x) = (x, x)$ regarded as a map of pairs $\Delta: (M, \partial M) \to (M \times M, \partial M \times M)$.

The homology class Δ_M can be expressed in terms of a basis for homology and Poincaré duality. Let $\{b_j^k\}$, $\{\bar{b}_j^k\}$ and $\{d_j^{n-k}\}$ be as in the discussion preceding Proposition 9.4. Let $a_j^{n-k} \equiv \bar{b}_j^k \cap [M] \in H_{n-k}(M, \partial M)$, $j = 1, \ldots, N(k)$.

PROPOSITION 9.6. $\Delta_M = \sum_{k \geqslant 0} \sum_{i=1}^{N(k)} (-1)^{k(n-k)} a_i^{n-k} \times b_i^k$. □

Proof. Without loss of generality, we can assume that M is connected. Observe

$$\begin{aligned} d_i^{n-k} \cap a_j^{n-k} &= d_i^{n-k} \cap (\bar{b}_j^k \cap [M]) = (d_i^{n-k} \cup \bar{b}_j^k) \cap [M] \\ &= (-1)^{(n-k)k} (\bar{b}_j^k \cup d_i^{n-k}) \cap [M] \\ &= (-1)^{(n-k)k} \bar{b}_j^k \cap (d_i^{n-k} \cap [M]) = (-1)^{(n-k)k} \bar{b}_j^k \cap b_i^k \\ &= (-1)^{(n-k)k} \delta_{ij} b_1^0 \end{aligned}$$

where δ_{ij} is Kronecker's delta.

By the Künneth formula, we can write

$$\Delta_M = \sum_{k \geq 0} \sum_{i=1}^{N(k)} \sum_{j=1}^{N(k)} c_{ij}^k a_i^{n-k} \times b_j^k$$

where $c_{ij}^k \in \mathbf{F}$. We have

$$\begin{aligned} (d_r^{n-l} \times \bar{b}_s^l) \cap \Delta_M &= (d_r^{n-l} \times \bar{b}_s^l) \cap \Delta_*([M]) = \Delta_*(\Delta^*(d_r^{n-l} \times \bar{b}_s^l) \cap [M]) \\ &= \Delta_*((d_r^{n-l} \cup \bar{b}_s^l) \cap [M]) \\ &= (-1)^{(n-l)l} \Delta_*((\bar{b}_s^l \cup d_r^{n-l}) \cap [M]) \\ &= (-1)^{(n-l)l} \Delta_*(\bar{b}_s^l \cap (d_r^{n-l} \cap [M])) \\ &= (-1)^{(n-l)l} \Delta_*(\bar{b}_s^l \cap b_r^l) = (-1)^{(n-l)l} \delta_{rs} b_1^0 \times b_1^0 . \end{aligned}$$

Now, $(d_r^{n-l} \times \bar{b}_s^l) \cap (a_i^{n-k} \times b_j^k) = 0$ whenever $l \neq k$ and

$$\begin{aligned} (d_r^{n-l} \times \bar{b}_s^l) \cap (a_i^{n-l} \times b_j^l) &= (-1)^{l(n-l)} (d_r^{n-l} \cap a_i^{n-l}) \times (\bar{b}_s^l \cap b_j^l) \\ &= \delta_{ri} \delta_{sj} b_1^0 \times b_1^0 . \end{aligned}$$

It follows that $c_{rs}^l = (-1)^{l(n-l)} \delta_{rs}$. □

Up to sign, the diagonal homology and the diagonal cohomology classes are Poincaré dual:

PROPOSITION 9.7. $D_M \cap ([M] \times [M]) = (-1)^n \Delta_M$.

Proof. Observe that

$$\begin{aligned} (b_i^k \times d_i^{n-k}) \cap ([M] \times [M]) &= (-1)^{(n-k)k} (b_i^k \cap [M]) \times (d_i^{n-k} \cap [M]) \\ &= (-1)^{(n-k)k} a_i^{n-k} \times b_i^k . \end{aligned}$$

Using the formula for D_M given by Proposition 9.2,

$$D_M \cap ([M] \times [M]) = \sum_{k \geqslant 0} (-1)^k \sum_{i=1}^{N(k)} (\bar{b}_i^k \times d_i^{n-k}) \cap ([M] \times [M])$$

$$= \sum_{k \geqslant 0} (-1)^k \sum_{i=1}^{N(k)} (-1)^{(n-k)k} a_i^{n-k} \times b_i^k$$

$$= (-1)^n \sum_{k \geqslant 0} \sum_{i=1}^{N(k)} (-1)^{k(n-k)} a_i^{n-k} \times b_i^k$$

$$= (-1)^n \Delta_M \quad \text{by Proposition 9.6.} \quad \square$$

Until now Y has been an arbitrary parameter space. For what follows we assume that Y is a closed q-dimensional manifold which is oriented over $\mathbf{F}$. Let $[Y] \in H_n(Y)$ be the fundamental class. Define $\mathrm{Gr}(p): M \times Y \to M \times Y \times M$ and $\mathrm{Gr}(F): M \times Y \to M \times Y \times M$ by $\mathrm{Gr}(p)(m, y) = (m, y, m)$ and $\mathrm{Gr}(F)(m, y) = (m, y, F(m, y))$. Define homology classes

$$A = \mathrm{Gr}(p)_*([M] \times [Y]) \in H_{n+q}(M \times Y \times M, M \times Y \times \partial M),$$
$$B = \mathrm{Gr}(F)_*([M] \times [Y]) \in H_{n+q}(M \times Y \times M, \partial M \times Y \times M).$$

We define the *intersection product* $A \bullet B \in H_q(M \times Y \times M)$ as follows. Let

$$\delta_1: H^n(M \times Y \times M, \partial M \times Y \times M) \to H_{n+q}(M \times Y \times M, M \times Y \times \partial M)$$
$$\delta_2: H^n(M \times Y \times M, M \times Y \times \partial M) \to H_{n+q}(M \times Y \times M, \partial M \times Y \times M)$$

be the Poincaré duality isomorphisms for the manifold triad $(M \times Y \times M; M \times Y \times \partial M, \partial M \times Y \times M)$ given by cap product with $[M] \times [Y] \times [M]$. Then

$$A \bullet B \equiv (\delta_1^{-1}(B) \cup \delta_2^{-1}(A)) \cap [M] \times [Y] \times [M].$$

Definition 9.8. The *graph intersection invariant of* F is $\theta'(F) \equiv (p_1)_*(A \bullet B) \in H_q(M)$ where $p_1: M \times Y \times M \to M$ is projection to the first M factor.

Remark 9.9. The graph intersection invariant of F can be obtained geometrically using transversality. Suppose F has no fixed points on $\partial M \times Y$. Then the boundaries of the embedded submanifolds $\mathrm{Gr}(p)(M \times Y) \subset M \times Y \times M$ and $\mathrm{Gr}(F)(M \times Y) \subset M \times Y \times M$ are disjoint and so these submanifolds may be made transverse via an ambient isotopy of the identity which leaves a neighborhood of the boundary of $M \times Y \times M$ (pointwise) fixed. The set theoretic intersection of the perturbed submanifolds is a closed orientable manifold of dimension q which we orient using the "intersection orientation" taken in the order: the perturbed $\mathrm{Gr}(p)(Y \times M)$ first followed by the perturbed $\mathrm{Gr}(F)(Y \times M)$. By Proposition 11.13 of [D_2, §VIII], the resulting oriented manifold is a cycle representing $A \bullet B$. Projecting this cycle to M via p_1 yields a representative of $\theta'(F)$.

The isomorphisms δ_1^{-1} and δ_2^{-1} can be described explicitly using the slant product. Let $(Z; \partial_1 Z, \partial_2 Z)$ be a compact oriented manifold triad and $K = \partial\partial_1 Z = \partial\partial_2 Z$. Since

$$(Z - \partial_1 Z, \partial_2 Z - K) \times (Z - \partial_2 Z, \partial_1 Z - K) \subset (Z \times Z, Z \times Z - \Delta)\,,$$
$$(Z - \partial_2 Z, \partial_1 Z - K) \times (Z - \partial_1 Z, \partial_2 Z - K) \subset (Z \times Z, Z \times Z - \Delta)$$

there are slant product pairings:

$$H^i(Z \times Z, Z \times Z - \Delta) \otimes H_j(Z - \partial_2 Z, \partial_1 Z - K) \xrightarrow{/} H^{i-j}(Z - \partial_1 Z, \partial_2 Z - K)$$
$$H^i(Z \times Z, Z \times Z - \Delta) \otimes H_j(Z - \partial_1 Z, \partial_2 Z - K) \xrightarrow{/} H^{i-j}(Z - \partial_2 Z, \partial_1 Z - K)$$

By the existence of collars, the inclusions $(Z - \partial_2 Z, \partial_1 Z - K) \hookrightarrow (Z, \partial_1 Z)$ and $(Z - \partial_1 Z, \partial_2 Z - K) \hookrightarrow (Z, \partial_2 Z)$ are homotopy equivalences and so we obtain pairings:

$$H^i(Z \times Z, Z \times Z - \Delta) \otimes H_j(Z, \partial_1 Z) \xrightarrow{/} H^{i-j}(Z, \partial_2 Z)\,,$$
$$H^i(Z \times Z, Z \times Z - \Delta) \otimes H_j(Z, \partial_2 Z) \xrightarrow{/} H^{i-j}(Z, \partial_1 Z)\,.$$

Let $m = \dim Z$. The inverse to the Poincaré duality isomorphisms

$$\delta_1 : H^{m-j}(Z, \partial_2 Z) \to H_j(Z, \partial_1 Z), \quad \delta_1(x) = x \cap ([Z] \times [Z])$$
$$\delta_2 : H^{m-j}(Z, \partial_1 Z) \to H_j(Z, \partial_2 Z), \quad \delta_2(x) = x \cap ([Z] \times [Z])$$

are explicitly given by $\delta_1^{-1}(y) = (-1)^{m(m-j)} T_Z/y$ and $\delta_2^{-1}(y) = (-1)^{m(m-j)} T_Z/y$ where $T_Z \in H^m(Z \times Z, Z \times Z - \Delta)$ is the Thom class of Z (see [MS, p. 135]).

PROPOSITION 9.10. $\theta'(F) = \bar{I}(F)([Y])$.

Proof. Without loss of generality, we may assume Y is connected. Let $S: M \times M \to M \times M$ be the "interchange map", i.e. $S(x, y) = (y, x)$. Now $S_*([M] \times [M]) = (-1)^n [M] \times [M]$ and so by Proposition 9.7, $\Delta_M = D_M \cap S_*([M] \times [M]) = S_*(S^*(D_M) \cap ([M] \times [M]))$. Hence $S_*(\Delta_M) = S^*(D_M) \cap ([M] \times [M])$. Using the inverse to the Poincaré duality isomorphism, we have $T_{M \times M}/S_*(\Delta_M) = S^*(D_M)$.

Define $\hat{F} = S \circ \bar{F}$. Then $\bar{F} = S \circ \hat{F}$. Also note that $p = p_1' \circ \hat{F}$ where $p_1' : M \times M \to M$ is projection to the first factor. From the definition of $\bar{I}(F)$,

$$
\begin{aligned}
\bar{I}(F)([Y]) &= p_*(\bar{F}^*(D_M) \cap ([M] \times [Y])) \\
&= (p_1')_* \hat{F}_* (\hat{F}^* S^*(D_M) \cap ([M] \times [Y])) \\
&= (p_1')_* (S^*(D_M) \cap \hat{F}_*([M] \times [Y])) \qquad (9.11) \\
&= (p_1')_* (T_{M\times M}/S_*(\Delta_M) \cap \hat{F}_*([M] \times [Y])) \\
&= (p_1'')_* (T_{M\times M} \cap \hat{F}_*([M] \times [Y]) \times S_*(\Delta_M))
\end{aligned}
$$

where p_1'' is projection to the first "M" factor.

Let $I': M \times Y \times M \to M \times M \times Y$ and

$$I'': M \times M \times Y \times M \times M \times Y \to M \times M \times M \times M \times Y \times Y$$

be the "interchange maps" given by $I'(m_1, y, m_2) = (m_1, m_2, y)$ and $I(m_1, m_2, y, m_3, m_4, y') = (m_1, m_2, m_3, m_4, y, y')$. Let $I = I'' \circ (I' \times I')$. Then $I^*(T_{M\times M} \times T_Y) = T_{M\times Y\times M}$ and

$$
\begin{aligned}
\theta'(F) &= (p_1)_*((\delta_1^{-1}(B) \cup \delta_2^{-1}(A)) \cap ([M] \times [Y] \times [M])) \\
&= (p_1)_*(\delta_1^{-1}(B) \cap (\delta_2^{-1}(A) \cap ([M] \times [Y] \times [M]))) \\
&= (p_1)_*(\delta_1^{-1}(B) \cap A) \\
&= (-1)^{qn}(p_1)_*((T_{M\times Y\times M}/B) \cap A) \qquad (9.12) \\
&= (-1)^{qn}(p_1^3)_*(T_{M\times Y\times M} \cap (A \times B)) \\
&= (-1)^{qn}(p_1^4)_* I_*(I^*(T_{M\times M} \times T_Y) \cap (A \times B)) \\
&= (-1)^{qn}(p_1^4)_*(T_{M\times M} \times T_Y \cap I_*(A \times B))
\end{aligned}
$$

where p_1^4 and $p_1^3 = p_1^4 \circ I$ are projections to the first "M" factor. We have

$$
\begin{aligned}
I'_*(A) &= \mathrm{Gr}(p)_*([Y] \times [M]) = S_*(\Delta_M) \times [Y] \\
I'_*(B) &= \mathrm{Gr}(F)_*([Y] \times [M]) = \hat{F}_*([Y] \times [M]) \times [y_0] + \beta
\end{aligned}
$$

where $[y_0] \in H_0(Y)$ is represented by $y_0 \in Y$ and β is a finite sum of the form $\beta = \sum_i v_i \times u_i$ with $u_i \in H_{n_i}(Y)$, $n_i \geqslant 1$, and $v_i \in H_{n+q-n_i}(M \times M, \partial M \times M)$. It follows that:

$$
\begin{aligned}
I_*(A \times B) = (-1)^{q(n+q)} S_*(\Delta_M) \times \hat{F}_*([Y] \times [M]) \times [Y] \times [y_0] \\
+ \sum_i (-1)^{q(n+q-n_i)} S_*(\Delta_M) \times v_i \times [Y] \times u_i\,.
\end{aligned}
$$

Since $T_Y \cap ([Y] \times u_i)$ lies in homology of degree $n_i > 0$,

$$
\begin{aligned}
&(p_1^4)_*((T_{M\times M} \times T_Y) \cap (S_*(\Delta_M) \times v_i \times [Y] \times u_i)) \\
&= (-1)^{q(q-n_i)}(p_1^4)_*((T_{M\times M} \cap (S_*(\Delta_M) \times v_i)) \times (T_Y \cap ([Y] \times u_i))) = 0\,.
\end{aligned}
$$

Using (9.12),

$$\begin{aligned}\theta'(F) &= (-1)^{qn}(p_1^4)_*(T_{M\times M}\times T_Y\cap I_*(A\times B))\\ &= (-1)^{qn}(-1)^{q(n+q)}(p_1^4)_*((T_{M\times M}\times T_Y)\cap(S_*(\Delta_M)\times\hat{F}_*([Y]\\ &\quad\times[M])\times[Y]\times[y_0]))\\ &= (-1)^q(-1)^q(p_1^4)_*((T_{M\times M}\cap(S_*(\Delta_M)\times\hat{F}_*([Y]\times[M])))\\ &\quad\times(T_Y\cap([Y]\times[y_0])))\\ &= (p_1^4)_*((T_{M\times M}\cap(S_*(\Delta_M)\times\hat{F}_*([Y]\times[M])))\times([y_0]\times[y_0]))\\ &= (p_1'')_*(T_{M\times M}\cap(S_*(\Delta_M)\times\hat{F}_*([Y]\times[M])))\\ &= \bar{I}(F)([Y]) \quad \text{by (9.11).} \quad \square\end{aligned}$$

Combining Propositions 9.4 and 9.10 yields:

THEOREM 9.13 (Trace Formula). *The graph intersection invariant is given by:*

$$\theta'(F) = \sum_{k\geqslant 0}(-1)^k\sum_{j=1}^{N(k)}\bar{b}_j^k\cap F_*(b_j^k\times[Y]). \quad \square$$

Remark. It is easy to check that Theorem 9.13 remains valid over a principal ideal domain R in place of the coefficient field $\mathbf{F}$, provided we assume that $H_*(M;R)$ is a free R-module.

10. PROOFS OF THEOREMS 1.1 AND 1.5

In this section we prove Theorems 1.1 and 1.5 which assert the equivalence, under appropriate hypotheses, of the four definitions of the first order Euler characteristic introduced in §1.

Proof of Theorem 1.1 (ii). Let M be a compact connected oriented PL or smooth n-manifold with boundary (as well as being the underlying simplicial complex of a compatible triangulation). Using Definition A_1, we are to show that $\chi_1(M)(\gamma) = -\theta(\gamma)$; the case of other coefficient rings R will then follow immediately. Fattening if necessary, assume $n\geqslant 4$.

Let $J: M\times I\to M$ be a homotopy from id_M to a map j, such that the graph of $J|_{M\times[\frac{1}{2},1]}$ meets the graph of $p|_{M\times[\frac{1}{2},1]}$ transversely in $|\chi(M)|$ arcs; this can be achieved by classical techniques of cancelling unnecessary pairs of fixed points. Note that j will then have precisely $|\chi(M)|$ fixed points, all transverse and having the same fixed point index.