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CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

by Peter WINGREN

ABSTRACT. A real-valued continuous nowhere-differentiable function on [0, 1] is constructed. Its graph F is proved to have the following property. If B is a Borel subset of F and if the projection of B on [0, 1] has positive Lebesgue measure, then the Hausdorff dimension of B is two.

0. INTRODUCTION

In 1903 Takagi [TAK, p. 176] gave an extremely simple construction of a nowhere differentiable real-valued continuous function on [0, 1]. Takagi's construction is

$$T(x) = \sum_{p=0}^{\infty} 2^{-p} \operatorname{dist}(2^p x, \mathbf{Z}) \tag{1}$$

where each term is a scaled version of the sawtooth function

$$\operatorname{dist}(x, \mathbf{Z}) := \inf\{\,|x - y| : y \in \mathbf{Z}\}\,. \tag{2}$$

Later, in 1930, van der Waerden [WAE] gave a similar example, which de Rham [RHA], in 1957, improved to an example identical with Takagi's.

It follows from a proof of Mauldin and Williams [M-W, pp. 795-797] that the graph of the Takagi function has a σ-finite linear Hausdorff measure and hence is of Hausdorff dimension 1.

In 1937 Besicovitch and Ursell [B-U, p. 29] constructed for an arbitrary α, $1 < \alpha < 2$, a real-valued nowhere-differentiable function in $C[0, 1]$ with graph of Hausdorff dimension α. They too used the sawtooth function $\operatorname{dist}(x, \mathbf{Z})$ as a building block in their construction.

In this paper we construct a real valued continuous function $f(x), x \in [0, 1]$, whose graph has an optimal property with respect to Hausdorff dimension and measure.