Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 41 (1995)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

Autor: Wingren, Peter

Kapitel: 2. A Lemma about mass distribution AND SUCCESSIVE TRANSLATIONS

DOI: https://doi.org/10.5169/seals-61818

constructed by closed dyadic cubes. The graph of a real valued function $f \in C[0, 1]$ is denoted by graph (f). By a dyadic cube we mean a cube which is the Cartesian product of dyadic intervals. If Q is an arbitrary dyadic closed cube, then the band of type $\{(x, y) : (x, z) \in Q$ for some $z \in \mathbf{R}\}$ is called a dyadic band. In our construction the dyadic bands of width 2^{-2^p} play a special role. They are called bands of generation $p, p = 0, 1, 2, \ldots$.

Acknowledgement. We would like to thank the referee for helpful suggestions.

1. A Lemma about mass distribution

By a mass distribution on a subset A of $\mathbf{R}^2$ we mean a measure μ on A such that $0 < \mu(A) < \infty$.

LEMMA 1. *Let* f *be a real valued measurable function defined on* $[0, 1]$. *Then there is a mass distribution* μ *on* $F := \text{graph}(f)$ *such that*

1) *for any two subintervals* I *and* I' *of* $[0, 1]$, *with* $m(I) = m(I')$,

$$\mu(I \times \mathbf{R}) = \mu(I' \times \mathbf{R})$$

and

2) *if for two Borel sets* B_1 *and* B_2 *in* $[0, 1] \times \mathbf{R}$ *there exists* $(x_0, y_0) \in \mathbf{R}^2$ *such that*

$$B_1 \cap F + (x_0, y_0) = B_2 \cap F$$

then

$$\mu(B_1) = \mu(B_2)\,.$$

Proof. Let B be an arbitrary Borel set in $\mathbf{R}^2$. Define

$$\mu(B) = m(\tilde{f}^{-1}(B))\,. \tag{5}$$

Then it is obvious that μ is a mass distribution on graph(f) and 1) and 2) follow from the translation invariance of the Lebesgue measure.

2. A Lemma about mass distribution and successive translations

LEMMA 2. *Let* $g(y) \geqslant 0$ *and* $g(y) \in L^1(\mathbf{R})$. *If* I *is a finite interval and* d *is a positive real number then*

$$\int_I \sum_{n=-\infty}^{\infty} g(y - nd)\,dy < \left(1 + \operatorname{int} \frac{m(I)}{d}\right) \cdot \|g\|\,. \tag{6}$$

Proof. It suffices to assume that $I = [0, m(I)]$. The general case will then follow by a change of variables. If we use the notation $M = \operatorname{int}\frac{m(I)}{d}$ we get

$$\int_I \sum_{n=-\infty}^{+\infty} g(y-nd)\,dy \leqslant \sum_{m=0}^{M} \int_{m\cdot d}^{(m+1)d} \sum_{n=-\infty}^{+\infty} g(y-nd)\,dy$$

$$(7) \qquad = \sum_{m=0}^{M} \sum_{n=-\infty}^{+\infty} \int_{m\cdot d}^{(m+1)d} g(y-nd)\,dy = \sum_{m=0}^{M} \sum_{n=-\infty}^{+\infty} \int_{(m-n)d}^{(m-n+1)d} g(t)\,dt$$

$$= \sum_{m=0}^{M} \|g\| = \left(1 + \operatorname{int}\left(\frac{m(I)}{d}\right)\right) \|g\| .$$

3. Hausdorff measure, net measure and Hausdorff dimension

This section presents standard results and definitions; see for example [FAL1].

The α-dimensional Hausdorff measure of a subset A of $\mathbf{R}^n$ is defined by

$$(8) \qquad H^{\alpha}(A) = \lim_{\delta \to 0} \inf_{\{U_i\}} \sum_{i=1}^{\infty} |U_i|^{\alpha} ,$$

where $\{U_i\}_1^{\infty}$ is a covering of A with $|U_i| < \delta$, $i = 1, 2, \ldots$, and the infimum is taken over all such coverings. The unique number α_0 such that $\alpha < \alpha_0$ implies $H^{\alpha}(A) = +\infty$ and $\alpha_0 < \alpha$ implies $H^{\alpha}(A) = 0$ is by definition the Hausdorff dimension of A.

The net measure $M^{\alpha}(A)$ of A is defined similarly except that the coverings $\{U_i\}$ consist of closed dyadic cubes. It follows that there exists a constant $c_1 > 0$ such that

$$(9) \qquad c_1 M^{\alpha}(A) \leqslant H^{\alpha}(A) \leqslant M^{\alpha}(A) .$$

Since $M^{\alpha}(A)$ and $H^{\alpha}(A)$ must therefore yield identical dimensions for A it will suffice to work with dyadic cubes.

4. Mass distribution and Hausdorff dimension

The following well known (see e.g. [FAL2, p. 232]) mass distribution principle will be used in Section 5.