

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 41 (1995)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CHAOTIC GROUP ACTIONS
Autor: Cairns, G. / Davis, G. / Elton, D. / Kolganova, A. / Perversi, P.
Kapitel: 5. Other Questions
DOI: <https://doi.org/10.5169/seals-61820>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 30.01.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Example. The group $G = \mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ acts faithfully and chaotically on \mathbf{T}^2 in such a way that none of the elements of G act chaotically on \mathbf{T}^2 in R. Devaney's sense.

Proof. First, as described above, there exist chaotic homeomorphisms of the closed disc (and hence of the closed square) which are the identity on the boundary. Let f be such a homeomorphism. Now consider \mathbf{T}^2 as the unit square with vertices (i, j) with $i, j \in \{0, 1\}$ and with edges identified in the usual manner; that is $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. Now use the x and y axes to subdivide \mathbf{T}^2 into 4 isometric subsquares. Let F be the homeomorphism of \mathbf{T}^2 obtained by applying f in each of the 4 subsquares. Let $g: \mathbf{T}^2 \rightarrow \mathbf{T}^2$ be the translation $g(x, y) = (x + 1/2, y)$. Similarly, define h by $h(x, y) = (x, y + 1/2)$. Then the group $G = \mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ generated by F, g and h acts chaotically on \mathbf{T}^2 . But clearly G contains no element which acts chaotically on \mathbf{T}^2 .

5. OTHER QUESTIONS

In this section we present some open questions which we have been unable to resolve. The main question is the following:

QUESTION 2. *Is there a faithful chaotic action of $\mathbf{Z} \times \mathbf{Z}$ on the torus \mathbf{T}^2 or the sphere \mathbf{S}^2 ?*

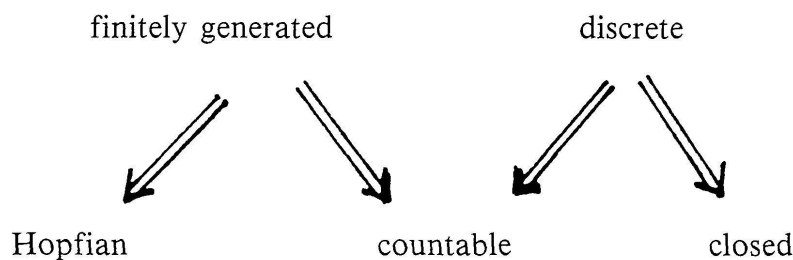
This question is of interest since in order to further the study of chaotic actions, one would naturally look to actions, on low dimensional manifolds, of groups which are simple generalizations of \mathbf{Z} . Because of Theorem 4, the obvious place to start is in dimension 2. Now the group $\mathbf{Z} \times \mathbf{Z} (= \mathbf{Z}^2)$ acts chaotically on \mathbf{T}^4 . But it is not clear whether \mathbf{Z}^2 acts chaotically and faithfully on \mathbf{T}^2 . Notice that $SL(2, \mathbf{Z})$ has no subgroup isomorphic to \mathbf{Z}^2 . Indeed, $PSL(2, \mathbf{Z})$ is a free product $\mathbf{Z}_2 * \mathbf{Z}_3$ (see [MKS]) and hence by Kurosh's theorem (see [LS]), it cannot have \mathbf{Z}^2 as a subgroup. But $PSL(2, \mathbf{Z})$ is the quotient of $SL(2, \mathbf{Z})$ by the group $\{\pm \text{Id}\} \cong \mathbf{Z}_2$. So $SL(2, \mathbf{Z})$ cannot have \mathbf{Z}^2 as a subgroup either.

It follows from the above discussion that if $G = \mathbf{Z}^2$ acts chaotically and faithfully on \mathbf{T}^2 , then G cannot contain a linear hyperbolic toral automorphism. Indeed, according to [AdPa], if f is a linear hyperbolic toral automorphism and if g is a homeomorphism of \mathbf{T}^n which commutes with f , then g is also a linear toral automorphism. (For more on commuting diffeomorphisms of tori, see [KaSp].)

Another general question is:

QUESTION 3. *What residually finite groups have a faithful chaotic action on some smooth connected compact manifold?*

Clearly finite groups are residually finite but have no faithful chaotic actions on any connected compact manifold. On the other hand, if a group G acts faithfully and chaotically on a compact manifold, then is G necessarily countable?, finitely generated?, discrete as a subgroup of $\text{Hom}(M)$?, closed as a subgroup of $\text{Hom}(M)$? These properties hold for the known examples of chaotic actions constructed from the action of $SL(n, \mathbf{Z})$ on \mathbf{T}^n . The properties would seem unlikely to hold in general, but counterexamples have proved to be elusive. Notice that for a smooth compact manifold M , a discrete subgroup $G \leq \text{Hom}(M)$ is necessarily countable, since $\text{Hom}(M)$ is second countable. So on smooth compact manifolds one has the following implications:



Notice that there is a simple partial result: Every topological group acting continuously, faithfully and chaotically on a Hausdorff space is totally pathwise disconnected. To see this, notice that if $G \subseteq \text{Hom}(M)$ acts chaotically, then the only continuous paths in G are the constant paths. Indeed, if γ_t is a path in G and if x has finite orbit under G , then as $\gamma_t(x)$ is a continuous path in M and as $\gamma_t(x)$ belongs to the (finite) orbit of x , so $\gamma_t(x)$ is independent of t . (We remark in passing that it is easy to see that every manifold admits a non-discrete totally pathwise disconnected group of homeomorphisms.)

REFERENCES

- [AdPa] ADLER, R. and R. PALAIS. Homeomorphic conjugacy of automorphisms on the torus. *Proc. Amer. Math. Soc.* 16 (1965), 1222-1225.
- [BBCDS] BANKS, J., J. BROOKS, G. CAIRNS, G. DAVIS and P. STACEY. On Devaney's definition of chaos. *Amer. Math. Monthly* 99 (1992), 332-334.
- [BFK] BRIN, M.I., J. FELDMAN and A. KATOK. Bernoulli diffeomorphisms and group extensions of dynamical systems with non-zero characteristic exponents. *Ann. of Math.* 113 (1981), 159-179.