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## 4. THE LEFSCHETZ THEOREM FOR FOLIATED MANIFOLDS

Let  $M$  be a compact  $m$  dimensional manifold and  $F$  a dimension  $n$  foliation on  $M$ . Then  $F$  is an  $n$  dimensional subbundle of  $TM$  such that for any two sections  $X, Y \in C^\infty(F)$ ,  $[X, Y] \in C^\infty(F)$ . The Frobenius Theorem says that for each  $x \in M$ , there is a neighborhood  $U$  of  $x$  and a diffeomorphism

$$\phi : \mathbf{R}^n \times \mathbf{R}^q \rightarrow U \quad n + q = m$$

so that for all  $z \in \mathbf{R}^n \times \mathbf{R}^q$ .

$$d\phi(T\mathbf{R}_z^n) = F_{\phi(z)}.$$

Such a  $(U, \phi)$  is called a foliation chart. Given  $x \in \mathbf{R}^q$ , the submanifold  $\phi(\mathbf{R}^n \times \{x\})$  is called a plaque, and is denoted  $P_x^U$ . It is a local integral submanifold of  $F$ . The submanifold  $\phi(\{0\} \times \mathbf{R}^q)$  is denoted  $\mathbf{R}_U^q$  and is called the transverse submanifold of  $(U, \phi)$ .

A leaf  $L$  of  $F$  is a maximal integral (i.e.  $TL_x = F_x$  for all  $x \in L$ ) submanifold of  $M$ . Thus  $\dim L = n$ . The Frobenius Theorem implies that through each point  $x$  in  $M$ , there passes a unique leaf, denoted  $L_x$ . Each leaf is a complete manifold of bounded geometry and the bounds are uniform for all leaves.

We now extend the Lefschetz Theorem for compact manifolds to a Lefschetz Theorem for foliations of a compact manifold. This is joint work with Connor Lazarov [HL 1]. In fact, we show how to improve the results of [HL 1] by removing the assumption that  $F$  admits a transverse invariant metric. For a K-theory version of this result, see the thesis of M-T. Benameur, [Be].

Choose a smooth metric on  $M$ . This induces a smooth metric on each leaf  $L$ , and  $L$  is complete with respect to this metric. Two different metrics on  $M$  induce quasi-isometric metrics on  $L$ .

## HAEFLIGER FORMS

Let  $\{U_i\}$  be a finite cover of  $M$  by foliation charts. For  $x \in U_i$ , denote its plaque by  $P_x^i$ . If  $U_i \cap U_j \neq \emptyset$  we define a local diffeomorphism  $f_{ij}$  from  $\mathbf{R}_{U_i}^q$  (hereafter denoted  $\mathbf{R}_i^q$ ) to  $\mathbf{R}_j^q$  as follows:

$$f_{ij}(x) = y \text{ if and only if } P_x^i \cap P_y^j \neq \emptyset.$$

The  $f_{ij}$  generate the holonomy pseudogroup, denoted  $H$ , which acts on the transversal space  $T = \cup_i \mathbf{R}_i^q$ . We may (and do) assume that the  $\mathbf{R}_i^q$  are disjoint.

Recall the following construction due to Haefliger [Ha]. Let  $\Omega_c^k(T)$  be the space of bounded measurable complex valued  $k$  forms on  $T$  with compact support. Denote by  $\Omega_c^k(T/H)$  the quotient of  $\Omega_c^k(T)$  by the vector subspace generated by elements of the form  $\alpha - h^*\alpha$  where  $h \in H$  and  $\alpha \in \Omega_c^k(T)$  has support contained in the range of  $h$ . Give  $\Omega_c^k(T/H)$  the quotient topology of the usual sup norm topology on  $\Omega_c^k(T)$ . Note that  $\Omega_c^k(T/H)$  does not depend of the choice of cover used to define it.

Denote by  $\Omega^{p+k}(M)$  the space of bounded measurable complex valued  $p+k$  forms on  $M$ . As the bundle  $TF$  is oriented, there is a continuous open surjective linear map,

$$\int_F : \Omega^{p+k}(M) \rightarrow \Omega_c^k(T/H).$$

It is given as follows. Let  $\omega \in \Omega^{p+k}(M)$  and let  $\{\psi_i\}$  be a partition of unity subordinate to the cover  $\{U_i\}$ . Set  $\omega_i = \psi_i\omega$ . We may integrate  $\omega_i$  along the fibers of the submersion  $\pi_i : U_i \rightarrow \mathbf{R}_i^q$  to obtain  $\bar{\omega}_i \in \Omega_c^k(\mathbf{R}_i^q)$ . Define  $\int_F \omega$  to be the class of  $\sum \bar{\omega}_i$  in  $\Omega_c^k(T/H)$ . It is independent of the choices made in defining it.

DIFFERENTIAL COMPLEXES ON  $M$  ELLIPTIC ALONG  $F$

A differential complex on  $M$  along  $F$  consists of :

- a) a finite collection of finite dimensional complex vector bundles  $E_0, \dots, E_k$  over  $M$
- b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \rightarrow C^\infty(E_{i+1})$$

with  $d_{i+1} \cdot d_i = 0$

- c) each  $d_i$  differentiates only in leaf directions.

For the sake of simplicity we assume that each  $d_i$  is first order.

Each of the classical complexes mentioned above (de Rham, Dolbeault, Signature and Twisted Spin) gives a leafwise complex on  $M$  provided that the leaves have the required structures and that these structures are coherent from leaf to leaf (i.e. come from a global structure on  $M$ ). For example, in the twisted Spin case, we require that the Spin structure on the leaves comes from a principal Spin( $n$ ) bundle  $P$  over  $M$  with  $P \times_{\text{Spin}(n)} \mathbf{R}^n \simeq TF$ , and that the leafwise auxiliary twisting bundle come from a bundle over  $M$ .

For a fixed leaf  $L$ , denote  $E_i|_L$  by  $E_i^L$  and by  $C_0^\infty(E_i^L)$  the space of smooth sections of  $E_i^L$  with compact support. The operator  $d_i$  induces one, denoted also by  $d_i$ ,

$$d_i : C_0^\infty(E_i^L) \rightarrow C_0^\infty(E_{i+1}^L)$$

and on  $L$  we have the complex

$$0 \rightarrow C_0^\infty(E_0^L) \xrightarrow{d_0} C_0^\infty(E_1^L) \xrightarrow{d_1} \dots \xrightarrow{d_{k-1}} C_0^\infty(E_k^L) \rightarrow 0.$$

We say that the complex  $(E, d)$  is elliptic along  $F$  provided that for each leaf  $L$ , the above complex is elliptic. We assume that  $(E, d)$  is elliptic along  $F$ .

### $L^2$ COHOMOLOGY OF $(E, d)$

Choose a smooth Hermitian metric on each bundle  $E_i$  over  $M$ . These induce metrics on each  $E_i^L$  and these metrics are unique up to quasi-isometry. Using these metrics we construct  $d_i^* : C_0^\infty(E_{i+1}^L) \rightarrow C_0^\infty(E_i^L)$  just as we did before. We then construct

$$\Delta_i^L : C_0^\infty(E_i^L) \rightarrow C_0^\infty(E_i^L)$$

and we extend  $\Delta_i$  to

$$\Delta_i^L : L^2(E_i^L) \rightarrow L^2(E_i^L)$$

just as before.

DEFINITION 4.1. *The  $i$ th  $L^2$  cohomology of  $(E, d)$  along the leaf  $L$ , denoted  $H_L^i(E, d)$  is*

$$H_L^i(E, d) = \ker \Delta_i^L.$$

*The  $i$ th  $L^2$  cohomology of  $(E, d)$  is denoted  $H^i(E, d)$  and it assigns to each leaf  $L$  the  $i$ th cohomology of  $(E, d)$  along  $L$ ,  $H_L^i(E, d)$ .*

### SOME FACTS

1.  $H_L^i(E, d)$  consists of smooth sections and  $\dim_{\mathbb{C}} H_L^i(E, d)$  may be infinite but is always countable.
2.  $\pi_L^i$ , the projection of  $L^2(E_i^L)$  onto  $H_L^i(E, d)$ , is a smoothing operator (on  $L$ ) with smooth Schwartz kernel  $k_L^i(x, y)$ .
3.  $k_L^i(x, y)$  is measurable as a function of  $L$  and bounded independently of  $L$ . In particular,  $\text{tr } k_L^i(x, x)$  is a bounded measurable function on  $M$  whose restriction to each leaf  $L$  is smooth.
4. Because of 3. above, we may define the dimension of  $H^i(E, d)$  to be the zero dimensional Haefliger form

$$\dim(H^i(E, d)) = \int_F \text{tr}(k_L^i(x, x)) dx,$$

where for any leaf  $L$  we denote the volume form obtained from the metric on  $L$  by  $dx$ . We may also define the Euler class of  $(E, d)$  as

$$\chi(E, d) = \sum_{i=0}^k (-1)^i \dim H^i(E, d).$$

GEOMETRIC ENDOMORPHISMS

Let  $f : M \rightarrow M$  be a diffeomorphism and assume that for each leaf  $L$  of  $F$ ,  $f(L) \subset L$ . For each  $i$ , let

$$A_i : f^* E_i \rightarrow E_i$$

be a smooth bundle map. We assume that  $T_i : C^\infty(E_i) \rightarrow C^\infty(E_i)$  where  $(T_i s)(x) = A_{i,x} s(f(x))$  satisfies

$$T_i d_{i-1} = d_{i-1} T_{i-1}.$$

The  $T_i$  then induce maps

$$T_i^L : C_0^\infty(E_i^L) \rightarrow C_0^\infty(E_i^L)$$

satisfying

$$T_i^L d_{i-1} = d_{i-1} T_{i-1}^L.$$

We call such a family  $T = (T_0, \dots, T_k)$  the geometric endomorphism of  $(E, d)$  defined by  $f$  and  $A = (A_0, \dots, A_k)$ . The  $T_i^L$  extend to uniformly bounded linear maps

$$T_i^L : L^2(E_i^L) \rightarrow L^2(E_i^L).$$

LEFSCHETZ NUMBER OF A GEOMETRIC ENDOMORPHISM

Set  $T_{i,L}^* = \pi_i^L \cdot T_i^L \cdot \pi_i^L$  and denote its Schwartz kernel by  $k_L^{T_i^*}(x, y)$ . Then  $k_L^{T_i^*}(x, y)$  is globally bounded, smooth on  $L \times L$ , and measurable. Thus  $\text{tr}(k_L^{T_i^*}(x, x))$  is a bounded measurable function on  $M$  which is smooth on each leaf  $L$ . We define the Lefschetz class of the geometric endomorphism  $T$  to be the Haefliger zero form

$$L(T) = \sum_{i=0}^k (-1)^i \int_F \text{tr}(k_L^{T_i^*}(x, x)) dx.$$

For our Lefschetz Theorem we shall also need two restrictions on the fixed point set,  $N$  of  $f$ . We require :

1.  $N = \bigcup_{\alpha} N_{\alpha}$  is a finite disjoint union of closed, connected submanifolds  $N_{\alpha}$ , each transverse to  $F$ .
2. for each  $x \in N \cap L = \bigcup_{\alpha} N_{\alpha}^L$  where  $N_{\alpha}^L = N_{\alpha} \cap L$ ,  $df_x$  has no eigen vector (in  $TL_x$ ) with eigenvalue  $+1$  in directions transverse (in  $L!$ ) to  $N_{\alpha}^L$ .

Note in particular that  $f = id_M$  satisfies these conditions.

#### FIXED POINT INDICES

Let  $\{U_i\}$  and  $\{\psi_i\}$  be as above. Suppose that for each  $L$  and  $\alpha$  we are given a differential form  $a_{\alpha}^L$  defined on  $N_{\alpha}^L$ . We define the Haefliger form  $\int_N a$  as

$$\int_N a = \sum_i \sum_{N_{\alpha}^L \cap P_x^i \neq \phi} \int_{N_{\alpha}^L \cap P_x^i} \psi_i a_{\alpha}^L.$$

Note that for any plaque  $P_x^i$ , only a finite number of  $N_{\alpha}^L$  satisfy  $N_{\alpha}^L \cap P_x^i \neq \phi$ . As  $\int_{N_{\alpha}^L \cap P_x^i} \psi_i a_{\alpha}^L$  is a differential form on the transversal  $\mathbf{R}_i^q$  of  $U_i$ , we may also consider it as a Haefliger form for  $F$ . As above, it is not difficult to show that the Haefliger form  $\int_N a$  does not depend on the choices made in defining it.

**THEOREM 4.2** (The Lefschetz Theorem for Foliations [HL 1]). *Let  $M$ ,  $F$ ,  $f$ ,  $T$ ,  $A$  and  $(E, d)$  be as above. To each  $N_{\alpha}^L \subset N$  we may associate a differential form  $a_{\alpha}^L$  which depends only on local data on  $N_{\alpha}^L$  so that*

$$L(T) = \int_N a.$$

The proof follows the outline given above for the classical case, done leafwise. There are some very formidable technical obstacles, but these can be overcome (see [HL 1]).

If  $(E, d)$  is the de Rham, Dolbeault, Signature or Twisted Spin complex of  $F$ , and  $f = id_M$ , and  $T = id$ , then  $a_j^L$  is the usual local integrand formula (computed on each leaf, not on  $M$ ) given by the Atiyah-Singer Index Theorem. We thus have an index theorem for foliated manifolds for these operators.

(Note that Connes has also proven an index theorem for foliated manifolds, (see [C]). As he works on the holonomy coverings of the leaves of  $F$ , his theorem is related to ours as the  $L^2$  covering index theorem is related to the ordinary index theorem.) If we take the codimension 0 foliation of  $M$  which has one leaf (namely  $M$ ), we recover the Atiyah-Singer Index Theorem for these operators. In general, i.e.  $f \neq id_M$ ,  $T = f^*$ ,  $a_j^L$  is the usual local integrand (computed on the fixed point set in each leaf, not in  $M$ ) given by the Atiyah-Singer  $G$  Index Theorem. If we take the codimension 0 foliation, we recover the Atiyah-Singer  $G$  Index Theorem and the Atiyah-Bott Lefschetz Theorem for these operators.

### 5. GROUP ACTIONS AND THE LEFSCHETZ THEOREM

Let  $F$  be an oriented  $2k$  dimensional foliation of a compact, oriented, Riemannian manifold  $M$ . Assume that  $F$  admits a  $Spin(2k)$  structure. That is, there is a principal  $Spin(2k)$  bundle  $P$  over  $M$  and an isomorphism of oriented bundles

$$P \times_{Spin(2k)} \mathbf{R}^{2k} \simeq TF.$$

We may then construct the bundles  $E^\pm = P \times_{Spin(2k)} \Delta^\pm$ . The leafwise Dirac operator  $D^+$  is constructed using the Riemannian structure on the leaves of  $F$  which is induced from  $M$ .

Let  $G$  be a compact, connected Lie group acting by isometries on  $M$ , taking each leaf of  $F$  to itself.  $G$  then acts on  $TF$ . We assume that  $G$  also acts on  $P$  (commuting with the action of  $Spin(2k)$ ) so that the induced action on  $P \times_{Spin(2k)} \mathbf{R}^{2k} \simeq TF$  is the given action on  $TF$ .  $G$  then acts on the bundles  $E^\pm$  and it commutes with the operator  $D^+$ , i.e.  $G$  is a group of geometric endomorphisms of the complex  $(E^\pm, D^+)$ .

Recall the  $\widehat{A}$  genus defined in Section 1.

DEFINITION 5.1. *The  $\widehat{A}$  genus of  $F$  is the Haefliger zero form*

$$\widehat{A}(F) = \int_F \widehat{A}_{k/2}(TF).$$

*In particular, if  $k$  is odd,  $\widehat{A}(F) = 0$ .*

Note that we have defined  $\widehat{A}(F)$  as the zero th order part of  $\int_F \widehat{A}(TF)$ .

For an interpretation of the higher order terms of  $\int_F \widehat{A}(TF)$ , see [He].