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Artikel: BARBIER'S THEOREM FOR THE SPHERE AND THE HYPERBOLIC PLANE

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3. Geodesic parallel coordinates

In this section we consider an oriented, connected, C^∞ surface S. For the sake of simplicity, we assume S is embedded in $\mathbf{R}^3$ whenever convenient (of course the hyperbolic plane cannot be embedded in $\mathbf{R}^3$, but our arguments have a local character, involving only the computation of derivatives; and there are surfaces in $\mathbf{R}^3$ which are locally isometric to the hyperbolic plane).

We consider a regular closed Jordan curve $\gamma(s)$ in S of constant width $\mathcal{W}$. If $\mathcal{L}$ is the perimeter of the curve, we extend $\gamma(s)$ periodically by setting $\gamma(s+\mathcal{L}) = \gamma(s)$.

We would like to say that the antipodal point $\tilde{p}$ of p is situated along the geodesic that cuts γ orthogonally at p. But some care is necessary, and we make an extra assumption on γ :

Standing Assumption (**SA**). *There exists $\varepsilon > 0$ such that, for every $p \in \gamma$, the restriction of the exponential map* $\exp_p$ *to*

$$\{v \in T_pS : \|v\| < \mathcal{W} + \varepsilon\}$$

is a diffeomorphism onto its image.

Condition **SA** ensures that there is exactly one minimizing geodesic between any two points of γ, and that γ is indeed the boundary of some Jordan region in S. On the hyperbolic plane, **SA** represents no restriction whatever, whereas on the sphere of radius $\frac{1}{\sqrt{K}}$ it is equivalent to the requirement that $\mathcal{W} < \frac{\pi}{\sqrt{K}}$ — and this is no strong restriction either, for in any case we would have $\mathcal{W} \le \frac{\pi}{\sqrt{K}}$, since the maximum (intrinsic) distance between distinct points on the sphere is $\frac{\pi}{\sqrt{K}}$.

Claim 1. *If a curve γ of constant width $\mathcal{W}$ satisfies* **SA** *then the minimizing geodesic between any pair of antipodal points p, $\tilde{p}$ intersects γ orthogonally at both p and $\tilde{p}$.*

Proof. Take a system of geodesic polar coordinates $\Phi(\rho,\theta)$ centered at $\tilde{p}$. If $\gamma(s_0) = p$ then there exists $\delta > 0$ such that, for $s \in]s_0-\delta, s_0+\delta[$, we can write $\gamma(s) = \Phi\big(\rho(s),\theta(s)\big)$ for some differentiable functions $\rho(s)$, $\theta(s)$. Our assumption implies that $\rho(s_0) = \mathcal{W}$ and $\rho(s) \le \mathcal{W}$ for all s, and therefore $\rho'(s_0) = 0$. Hence $\gamma'(s_0) = \theta'(s_0)\,\Phi_\theta$, which implies that γ and the radial (minimizing) geodesic from $\tilde{p}$ to p cut each other orthogonally at p. Reversing the roles of p and $\tilde{p}$ we show that the intersection at $\tilde{p}$ is also orthogonal. $\square$

We have just observed that if p, $\tilde{p}$ are antipodal points of γ then γ is inside the geodesic circle $\mathcal{C}(\tilde{p}, \mathcal{W})$ of centre $\tilde{p}$ and radius $\mathcal{W}$, and touches it at the point p. As in the euclidean case, the geodesic curvature of $\mathcal{C}(\tilde{p}, \mathcal{W})$ at p is a lower bound for the geodesic curvature of γ at p, as we now proceed to show. We assume that both curves are traversed counterclockwise (i.e., the Jordan region bounded by γ is always to our left as we move around γ), and recall that the coeficients E, F, G of the first fundamental form of $\Phi(\rho, \theta)$ are such that $E \equiv 1$ and $F \equiv 0$ (see [dC], p. 287).

CLAIM 2. *Let the coordinates of p be $\rho = \mathcal{W}$, $\theta = \theta_0$, and denote by $k_g(p)$ and $\tilde{k}_g(p)$ the geodesic curvatures at p of γ and $\mathcal{C}(\tilde{p}, \mathcal{W})$, respectively. Then we have*

$$k_g(p) \geq \tilde{k}_g(p) = \left. \frac{G_\rho}{2G} \right|_{(\mathcal{W}, \theta_0)} .$$

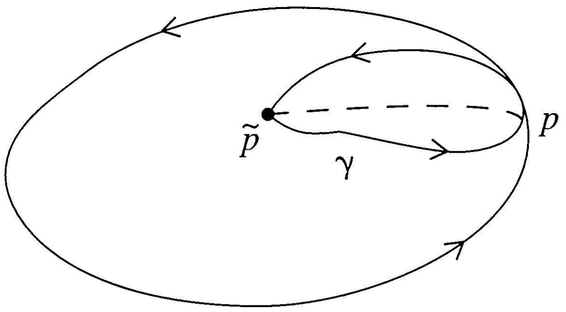

FIGURE 3

Proof. We can reparametrize γ in a neighbourhood of p by setting $\gamma(t) = \Phi\big(\rho(t), \theta_0 + t\big)$ for $t \in \,]-\delta, \delta[$. Thus $\rho(0) = \mathcal{W}$ and, as in the proof of Claim 1, $\rho(t)$ attains a maximum at $t = 0$, so that $\rho'(0) = 0$ and $\rho''(0) \leq 0$. The geodesic curvature of γ at $\gamma(t)$ is given by

$$k_g(t) = \frac{1}{\|\gamma'(t)\|^2} \big\langle \gamma''(t), n(t) \big\rangle ,$$

where $n(t)$ is the *unit* vector such that $\big(\gamma'(t), n(t)\big)$ is a positively oriented *orthogonal* basis of $T_{\gamma(t)}S$. We have $\gamma'(0) = \Phi_\theta$, and therefore $n(0) = -\Phi_\rho$, $\|\gamma'(0)\|^2 = G$, and

$$k_g(p) = k_g(0) = \frac{1}{G} \big\langle \gamma''(0), -\Phi_\rho \big\rangle .$$

Since $\gamma''(0) = \rho''(0)\, \Phi_\rho + \Phi_{\theta\theta}$, it follows that

$$k_g(p) = -\frac{1}{G}\rho''(0) - \frac{1}{G}\langle \Phi_{\theta\theta}, \Phi_\rho\rangle \geq -\frac{1}{G}\langle \Phi_{\theta\theta}, \Phi_\rho\rangle \cdot \tag{5}$$

Our calculations also show that the right-hand side of (5) is just $\tilde{k}_g(p)$. By taking the derivative with respect to θ of the equality $\langle \Phi_\theta, \Phi_\rho\rangle \equiv 0$, we obtain $\langle \Phi_{\theta\theta}, \Phi_\rho\rangle = -\frac{1}{2}G_\rho$ — and this, together with (5), proves our claim. $\square$

At this point we recall ([dC], p. 289) that in S_K the coefficient G is given by

$$\frac{1}{K}\sin^2(\sqrt{K}\rho) \text{ if } K > 0; \qquad \rho^2 \text{ if } K = 0; \qquad -\frac{1}{K}\sinh^2(\sqrt{-K}\rho) \text{ if } K < 0$$

— and thus in S_K Claim 2 reduces to:

The geodesic curvature $k_g(s)$ of a curve of constant width $\mathcal{W}$ in S_K is such that

$$k_g(s) \geq F(K, \mathcal{W}), \tag{6}$$

where $F(K,\mathcal{W})$ is given by

$$\frac{\sqrt{K}\cos(\sqrt{K}\,\mathcal{W})}{\sin(\sqrt{K}\,\mathcal{W})} \; \textit{if } K > 0; \quad \frac{1}{\mathcal{W}} \; \textit{if } K = 0; \quad \frac{\sqrt{-K}\cosh(\sqrt{-K}\,\mathcal{W})}{\sinh(\sqrt{-K}\,\mathcal{W})} \; \textit{if } K < 0 .$$

Notice that we do not necessarily have $k_g(s) \geq 0$: for $K > 0$ and $\mathcal{W} > \frac{\pi}{2\sqrt{K}}$, the lower bound in (6) is negative. Related to this is the fact that not all curves of constant width in the sphere are convex (see the remark at the end of this section).

Now we let $n(s)$ be the *unit* vector field along $\gamma(s)$ which is orthogonal to $\gamma'(s)$ and points to the interior of the region bounded by γ, so that $(\gamma'(s), n(s))$ is positively oriented. If we travel a distance $\mathcal{W}$ along the geodesic $t \mapsto \exp_{\gamma(s)}(t\,n(s))$ we reach the antipodal point $\Pi(\gamma(s))$ of $\gamma(s)$. In other words,

$$\Pi \circ \gamma(s) = \exp_{\gamma(s)}(\mathcal{W}\, n(s)) . \tag{7}$$

It is only natural to consider the map $\Psi(t,s) = \exp_{\gamma(s)}(-t\,n(s))$, where the minus sign ensures that (Ψ_t, Ψ_s) is positively oriented for small t. This is not really a parametrization, since it is not injective and may have singularities. We define the coefficients $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ by

$$\mathcal{E} = \langle \Psi_t, \Psi_t\rangle, \quad \mathcal{F} = \langle \Psi_t, \Psi_s\rangle, \quad \mathcal{G} = \langle \Psi_s, \Psi_s\rangle \cdot$$

CLAIM 3. *The following equalities hold:* $\mathcal{E} \equiv 1$; $\mathcal{F} \equiv 0$; $\mathcal{G}(0,s) = 1$ *for all* $s \in \mathbf{R}$.

Proof. For fixed s, the curve $t \mapsto \Psi(t,s)$ is a geodesic parametrized with constant speed $\|n(s)\| = 1$, and therefore $\mathcal{E} \equiv 1$. The third equality is obvious. To prove the second one, we observe that $\mathcal{F}(0,s) = \langle -n(s), \gamma'(s) \rangle = 0$ and that

$$\frac{\partial \mathcal{F}}{\partial t} = \langle \Psi_{tt}, \Psi_s \rangle + \langle \Psi_t, \Psi_{st} \rangle = \left\langle \frac{D\Psi_t}{\partial t}, \Psi_s \right\rangle + \frac{1}{2}\frac{\partial \mathcal{E}}{\partial s} = 0,$$

where $\dfrac{D\Psi_t}{\partial t}$ denotes the covariant derivative of the "velocity" vector field $t \mapsto \Psi_t(t,s)$ along the geodesic $t \mapsto \Psi(t,s)$ (which is identically zero by the definition of a geodesic). □

In the neighbourhood of any point (t,s) where $\mathcal{G}$ is non-zero, the map Ψ is a true parametrization, and by Claim 3 its coefficients $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are analogous to the coefficients E, F, G of the geodesic polar coordinates. Thus the proof of Claim 2 shows that, provided $\Psi(t,s)$ agrees with the orientation of S, the geodesic curvature of the curves $t = \text{constant}$ is given by

$$\frac{\mathcal{G}_t}{2\mathcal{G}} = \frac{(\sqrt{\mathcal{G}})_t}{\sqrt{\mathcal{G}}};$$

in particular, setting $t = 0$ and using Claim 3, we obtain

(8) $$(\sqrt{\mathcal{G}})_t(0,s) = k_g(s).$$

There exists a very useful formula for the Gaussian curvature K in terms of the coefficients of an orthogonal parametrization ([dC], p. 237), which in this case simplifies to

$$(\sqrt{\mathcal{G}})_{tt} + K\sqrt{\mathcal{G}} = 0.$$

This formula holds whenever $\mathcal{G}(t,s) \neq 0$. Turning our attention to S_K, $t \mapsto \sqrt{\mathcal{G}}(t,s)$ is then the solution of the differential equation

(9) $$x''(t) + K\,x(t) = 0$$

which, by Claim 3 and (8), satisfies the initial conditions $x(0) = 1$ and $x'(0) = k_g(s)$. Thus we find that $\sqrt{\mathcal{G}}(t,s)$ is given by:

(10) $$\cos(\sqrt{K}\,t) + \frac{k_g(s)}{\sqrt{K}}\sin(\sqrt{K}\,t) \qquad \text{if } K > 0,$$

(11) $$1 + t\,k_g(s) \qquad \text{if } K = 0,$$

(12) $$\cosh(\sqrt{-K}\,t) + \frac{k_g(s)}{\sqrt{-K}}\sinh(\sqrt{-K}\,t) \qquad \text{if } K < 0.$$

We are running into trouble here: formulas (10)-(12) may assume negative values for $t \neq 0$, and $\sqrt{\mathcal{G}}$ is necessarily non-negative. However, we must keep in mind that in any case $\mathcal{G}$ is a differentiable (C^∞) function, as its definition ensures. The only way to reconcile this with the fact that $t \mapsto \sqrt{\mathcal{G}}(t,s)$ is a solution of (9) whenever $\mathcal{G}(t,s) \neq 0$ is that $\mathcal{G}(t,s)$ be equal to the *square* of formulas (10)-(12) for all (t,s).

Let f be a lifting of the antipodal map Π as in Lemma E. We can rewrite (7) as

$$\gamma \circ f(s) = \Psi(-\mathcal{W}, s).$$

Taking the derivative of both sides we obtain $f'(s)\,\gamma'\big(f(s)\big) = \Psi_s|_{(-\mathcal{W},s)}$, and from here we get

$$[f'(s)]^2 = \mathcal{G}(-\mathcal{W}, s). \tag{13}$$

The reader should now check that inequality (6) yields that, for $t = -\mathcal{W}$, each of the formulas (10)-(12) is non-positive for all $s \in \mathbf{R}$. Since $f'(s) > 0$, (13) and the above discussion imply that $f'(s)$ is equal to

$$\frac{k_g(s)}{\sqrt{K}} \sin(\sqrt{K}\,\mathcal{W}) - \cos(\sqrt{K}\,\mathcal{W}) \qquad \text{if } K > 0, \tag{14}$$

$$\mathcal{W}\,k_g(s) - 1 \qquad \text{if } K = 0, \tag{15}$$

$$\frac{k_g(s)}{\sqrt{-K}} \sinh(\sqrt{-K}\,\mathcal{W}) - \cosh(\sqrt{-K}\,\mathcal{W}) \qquad \text{if } K < 0. \tag{16}$$

Formula (15) was already known from §2. In the next section we use formulas (14) and (16) to prove Theorem B and Corollary C. As an appetizer we now prove Theorem D.

Proof of Theorem D. This is a simple consequence of the uniqueness part of Lemma E. Under our hypothesis, a possible lifting of Π is $f(s) = s + \frac{1}{2}\mathcal{L}$, and therefore $f'(s) = 1$ for all $s \in \mathbf{R}$. Each of the formulas (14)-(16) then implies that the geodesic curvature k_g of γ is constant. Substituting the value of k_g in (10)-(12) we find that $\mathcal{G}(-\frac{1}{2}\mathcal{W}, s) = 0$ for all $s \in \mathbf{R}$. Hence $s \mapsto \Psi(-\frac{1}{2}\mathcal{W}, s)$ is constant, say equal to p, and therefore γ is the geodesic circle $\mathcal{C}(p, \frac{1}{2}\mathcal{W})$. $\square$

REMARK. We have so far excluded from our discussion curves of constant width $\frac{\pi}{\sqrt{K}}$ on the sphere $x^2+y^2+z^2=\frac{1}{K}$ $(K>0)$. Although our methods do not apply to these curves, they are easily dealt with, being characterized as the Jordan curves γ which remain invariant under the isometry $g\colon S_K \to S_K$ given by $g(x,y,z)=(-x,-y,-z)$. This map g exchanges the two regions bounded by γ in S_K (so these regions have the same area $\frac{2\pi}{K}$), and also exchanges the two arcs into which γ is divided by any pair of antipodal points (so these two arcs have the same length). Hence Theorem D is not valid in this case. If we consider (for small d) a parallel curve γ_d to γ then γ has constant width $\frac{\pi}{\sqrt{K}}-2d$. Since γ has arbitrarily long perimeter and does not need to be convex, the same applies to γ_d (but the longer the perimeter of γ, the smaller d must be in order to ensure that γ_d has no self-intersections).

4. Proof of the main results

We have now gathered all the necessary tools, and the proofs of Theorem B and Corollary C are a simple matter.

Proof of Theorem B. We assume $K>0$, the case $K<0$ being similar. Using (14) we have

$$\mathcal{L}=f(\mathcal{L})-f(0)=\int_0^{\mathcal{L}} f'(s)\,ds=\frac{\sin(\sqrt{K}\,\mathcal{W})}{\sqrt{K}}\int_0^{\mathcal{L}} k_g(s)\,ds-\mathcal{L}\cos(\sqrt{K}\,\mathcal{W}),$$

and therefore

$$\begin{aligned}\mathcal{L}&=\frac{\sin(\sqrt{K}\,\mathcal{W})}{\sqrt{K}\,\{1+\cos(\sqrt{K}\,\mathcal{W})\}}\int_0^{\mathcal{L}} k_g(s)\,ds\\&=\frac{1}{\sqrt{K}}\tan\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right)\{2\pi-K\mathcal{A}\}\end{aligned}$$

by the Gauss-Bonnet theorem. □

Proof of Corollary C. First we treat the case $K>0$. From Theorem B we see that $\mathcal{A}<\frac{2\pi}{K}$, which means that the region we are interested in has the smallest area of the two regions bounded by γ in S_K. We also assume