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Artikel: BARBIER'S THEOREM FOR THE SPHERE AND THE HYPERBOLIC PLANE

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Kapitel: 4. Proof of the main results

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REMARK. We have so far excluded from our discussion curves of constant width $\frac{\pi}{\sqrt{K}}$ on the sphere $x^2+y^2+z^2 = \frac{1}{K}$ $(K > 0)$. Although our methods do not apply to these curves, they are easily dealt with, being characterized as the Jordan curves γ which remain invariant under the isometry $g\colon S_K \to S_K$ given by $g(x,y,z) = (-x,-y,-z)$. This map g exchanges the two regions bounded by γ in S_K (so these regions have the same area $\frac{2\pi}{K}$), and also exchanges the two arcs into which γ is divided by any pair of antipodal points (so these two arcs have the same length). Hence Theorem D is not valid in this case. If we consider (for small d) a parallel curve γ_d to γ then γ has constant width $\frac{\pi}{\sqrt{K}} - 2d$. Since γ has arbitrarily long perimeter and does not need to be convex, the same applies to γ_d (but the longer the perimeter of γ, the smaller d must be in order to ensure that γ_d has no self-intersections).

4. PROOF OF THE MAIN RESULTS

We have now gathered all the necessary tools, and the proofs of Theorem B and Corollary C are a simple matter.

Proof of Theorem B. We assume $K > 0$, the case $K < 0$ being similar. Using (14) we have

$$\mathcal{L} = f(\mathcal{L}) - f(0) = \int_0^{\mathcal{L}} f'(s)\,ds = \frac{\sin(\sqrt{K}\,\mathcal{W})}{\sqrt{K}} \int_0^{\mathcal{L}} k_g(s)\,ds - \mathcal{L}\cos(\sqrt{K}\,\mathcal{W}),$$

and therefore

$$\begin{aligned}\mathcal{L} &= \frac{\sin(\sqrt{K}\,\mathcal{W})}{\sqrt{K}\,\{1+\cos(\sqrt{K}\,\mathcal{W})\}} \int_0^{\mathcal{L}} k_g(s)\,ds \\ &= \frac{1}{\sqrt{K}} \tan\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right) \{2\pi - K\,\mathcal{A}\}\end{aligned}$$

by the Gauss-Bonnet theorem. □

Proof of Corollary C. First we treat the case $K > 0$. From Theorem B we see that $\mathcal{A} < \frac{2\pi}{K}$, which means that the region we are interested in has the smallest area of the two regions bounded by γ in S_K. We also assume that $\mathcal{L} \leq \frac{2\pi}{\sqrt{K}}$, otherwise $\mathcal{L}$ is too large for γ to be a circle and the desired inequality holds trivially. Under these conditions inequality (1) is equivalent to

$$\mathcal{A} \leq \frac{1}{K}\left\{2\pi - \sqrt{4\pi^2 - K\mathcal{L}^2}\right\}. \tag{17}$$

Combining Theorem B and (17) we obtain

$$\mathcal{L} \geq \frac{1}{\sqrt{K}} \tan\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right)\sqrt{4\pi^2 - K\mathcal{L}^2},$$

which is equivalent to

$$\mathcal{L} \geq \frac{2\pi}{\sqrt{K}} \sin\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right) \tag{18}$$

— and this is the inequality we want. If equality holds in (18) then it also holds in each of the equivalent inequalities (17) and (1) — and therefore γ is a circle.

The case $K < 0$ has a similar (and easier) treatment. We begin by rewriting (1) in the form

$$\mathcal{A} \leq -\frac{1}{K}\left\{\sqrt{4\pi^2 - K\mathcal{L}^2} - 2\pi\right\},$$

and then proceed as before. □

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