

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: BARBIER'S THEOREM FOR THE SPHERE AND THE HYPERBOLIC PLANE
Autor: Araújo, Paulo Ventura
Kapitel: 4. Proof of the main results
DOI: <https://doi.org/10.5169/seals-87880>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 06.02.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

REMARK. We have so far excluded from our discussion curves of constant width $\frac{\pi}{\sqrt{K}}$ on the sphere $x^2 + y^2 + z^2 = \frac{1}{K}$ ($K > 0$). Although our methods do not apply to these curves, they are easily dealt with, being characterized as the Jordan curves γ which remain invariant under the isometry $g: S_K \rightarrow S_K$ given by $g(x, y, z) = (-x, -y, -z)$. This map g exchanges the two regions bounded by γ in S_K (so these regions have the same area $\frac{2\pi}{K}$), and also exchanges the two arcs into which γ is divided by any pair of antipodal points (so these two arcs have the same length). Hence Theorem D is not valid in this case. If we consider (for small d) a parallel curve γ_d to γ then γ has constant width $\frac{\pi}{\sqrt{K}} - 2d$. Since γ has arbitrarily long perimeter and does not need to be convex, the same applies to γ_d (but the longer the perimeter of γ , the smaller d must be in order to ensure that γ_d has no self-intersections).

4. PROOF OF THE MAIN RESULTS

We have now gathered all the necessary tools, and the proofs of Theorem B and Corollary C are a simple matter.

Proof of Theorem B. We assume $K > 0$, the case $K < 0$ being similar. Using (14) we have

$$\mathcal{L} = f(\mathcal{L}) - f(0) = \int_0^{\mathcal{L}} f'(s) ds = \frac{\sin(\sqrt{K} \mathcal{W})}{\sqrt{K}} \int_0^{\mathcal{L}} k_g(s) ds - \mathcal{L} \cos(\sqrt{K} \mathcal{W}),$$

and therefore

$$\begin{aligned} \mathcal{L} &= \frac{\sin(\sqrt{K} \mathcal{W})}{\sqrt{K} \{1 + \cos(\sqrt{K} \mathcal{W})\}} \int_0^{\mathcal{L}} k_g(s) ds \\ &= \frac{1}{\sqrt{K}} \tan\left(\frac{\sqrt{K} \mathcal{W}}{2}\right) \{2\pi - K \mathcal{A}\} \end{aligned}$$

by the Gauss-Bonnet theorem. \square

Proof of Corollary C. First we treat the case $K > 0$. From Theorem B we see that $\mathcal{A} < \frac{2\pi}{K}$, which means that the region we are interested in has the smallest area of the two regions bounded by γ in S_K . We also assume

that $\mathcal{L} \leq \frac{2\pi}{\sqrt{K}}$, otherwise \mathcal{L} is too large for γ to be a circle and the desired inequality holds trivially. Under these conditions inequality (1) is equivalent to

$$(17) \quad A \leq \frac{1}{K} \left\{ 2\pi - \sqrt{4\pi^2 - K \mathcal{L}^2} \right\}.$$

Combining Theorem B and (17) we obtain

$$\mathcal{L} \geq \frac{1}{\sqrt{K}} \tan\left(\frac{\sqrt{K} \mathcal{W}}{2}\right) \sqrt{4\pi^2 - K \mathcal{L}^2},$$

which is equivalent to

$$(18) \quad \mathcal{L} \geq \frac{2\pi}{\sqrt{K}} \sin\left(\frac{\sqrt{K} \mathcal{W}}{2}\right)$$

— and this is the inequality we want. If equality holds in (18) then it also holds in each of the equivalent inequalities (17) and (1) — and therefore γ is a circle.

The case $K < 0$ has a similar (and easier) treatment. We begin by rewriting (1) in the form

$$A \leq -\frac{1}{K} \left\{ \sqrt{4\pi^2 - K \mathcal{L}^2} - 2\pi \right\},$$

and then proceed as before. \square

REFERENCES

- [B] BARBIER, E. Note sur le problème de l'aiguille et le jeu du joint couvert. *J. Math. Pures Appl. (2)* 5 (1860), 273–286.
- [Bl] BLASCHKE, W. Einige Bemerkungen über Kurven und Flächen von konstanter Breite. *Ber. d. Verh. d. Sächs. Akad. Leipzig* 67 (1915), 290–297.
- [C] CADWELL, J.H. *Topics in Recreational Mathematics*. Cambridge University Press, 1966.
- [dC] DO CARMO, M.P. *Differential Geometry of Curves and Surfaces*. Prentice-Hall 1976.
- [E] EGGLESTON, H.G. *Convexity*. Cambridge University Press, 1958.
- [HS] HAMMER, P.C. and T.J. SMITH. Conditions equivalent to central symmetry of convex curves. *Proc. Cambridge Philos. Soc.* 60 (1964), 779–785.
- [O] OSSERMAN, R. Bonnesen-style isoperimetric inequalities. *Amer. Math. Monthly* 86 (1979), 1–29.