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Kapitel: 6. SUFFICIENCY IN THEOREM 3

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Now, proceeding inductively, consider the case of $k > 1$ homology classes. One of these homology classes, say α_k, has a minimal partition C_k, D_k, in the sense that C_k contains no other C_j or D_j for j< k. By the preceding argument we may assume that $A_k = A'_k$. One side of A_k contains no other simple closed curves A_j or A'_j. Excise this side to obtain a new planar surface H containing the remaining simple closed curves. By induction there is a homeomorphism h of H moving A_j onto A'_j for $1 \leq j \leq k-1$. and mapping each boundary curve into itself. We can then reinsert the excised domain to complete the argument.

The results of this section, with the exception of Theorem 5.6 above, hold *mutatis mutandi* for compact non-planar surfaces G with boundary, provided one only considers homology classes given as linear combinations of the classes represented by the boundary curves. Each such simple closed curve in the interior of G would separate G. Uniqueness, however, is obstructed by needing to know the genus of each complementary domain.

6. Sufficiency in Theorem 3

Let $S \subset H_1(F)$ denote a finite set of distinct homology classes satisfying the Intersection Condition, the Summand Condition, and the Size Condition of Theorem 1, which we wish to represent by pairwise disjoint simple closed curves. Suppose that the linear span of S has rank n and extract from S n elements $\alpha_1, \ldots, \alpha_n$ that form a basis for this span. Now each element γ_i in the remaining subset T of S can be expressed as a linear combination

$$\gamma_i = \sum_j \varepsilon_{ij} \alpha_j .$$

We refer to the γ_i as "composite classes."

Lemma 6.1. *Each coefficient* ε_{ij} *in the linear combination* $\gamma_i = \sum_j \varepsilon_{ij} \alpha_j$ *is* 1, -1, *or* 0.

Proof. To see this, consider the span of the set consisting of any one γ_i together with all α_k, $k \neq j$. Elementary change of basis operations show that this span is the same as the span of $\varepsilon_{ij} \alpha_j$ and the α_k, $k \neq j$. By the Summand Condition, this span must be a summand, and it therefore follows that $\varepsilon_{ij} = \pm 1$ or 0.

If card $T = m$, then the collection of all $\gamma_i \in T$ can be described by an m by n matrix M of 0's, 1's, and -1's. Then the proof of Lemma 6.1 extends to give the following consequence of the Summand Condition.

LEMMA 6.2. *Each square submatrix N of M has* $|\det N| \leq 1$.

Proof. Let N be a $k \times k$ submatrix. Up to relabeling we may assume that N consists of the ε_{ij}, $1 \leq i,j \leq k$. Now consider the span of $\gamma_1, \dots, \gamma_k, \alpha_{k+1}, \dots, \alpha_n$. On the one hand, the Summand Condition says that this span must be a direct summand. On the other hand, the span is the same as the span of $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k, \alpha_{k+1}, \dots, \alpha_n$, where $\tilde{\gamma}_i = \sum_{j \leq k} \varepsilon_{ij} \alpha_j$ is the projection of γ_i into the span of $\alpha_1, \dots, \alpha_k$. But this span is clearly the direct sum of the span of $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ and of $\alpha_{k+1}, \dots, \alpha_n$. It follows that the span of $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ is a direct summand of the span of $\alpha_1, \dots, \alpha_k$. Standard matrix theory then implies that the determinant of the matrix of coefficients of $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ is ± 1 or 0. But this matrix is the upper left matrix N.

In what follows here we will assume that F has genus n. By this we mean that there is a corresponding set of homology classes in a surface of genus n, and that we may view the given surface as being obtained from the genus n surface by attaching handles. It is clear that if the homology classes can be realized in the surface of genus n, then they can be realized in the given surface. The converse of this statement is also true, but less obvious. We will prove it in a subsequent section.

We may also assume that we have already represented elements $\alpha_1, \dots, \alpha_n$ by disjoint simple closed curves elements $A_1, \dots, A_n$, by Proposition 4.2. We attempt to represent the remaining classes in T, the complement of $\alpha_1, \dots, \alpha_n$ in S. Let $\widehat{F}$ denote F cut open along the A_i. Then $\widehat{F}$ is a 2-sphere with $2n$ holes, with orientable boundary consisting of one copy A_i^+ of each A_i and one copy A_i^- of each A_i with its orientation reversed.

By the results in Section 5 we understand completely when a family of homology classes in $\widehat{F}$ can be realized by pairwise disjoint simple closed curves. We need to see how to lift the classes in T to realizable class in $\widehat{F}$. Now $H_1(\widehat{F})$ is generated by the classes $[A_i^+]$ and $[A_i^-]$ subject to the single relation $\sum_i([A_i^+] + [A_i^-]) = 0$. The natural inclusion of $\widehat{F}$ in F induces a homomorphism $H_1(\widehat{F}) \to H_1(F)$ where $[A_i^+] \to [A_i]$ and $[A_i^-] \to -[A_i]$. This homomorphism maps surjectively onto the subgroup generated by $A_1, \dots, A_n$. Its kernel is generated by terms of the form $[A_i^+] + [A_i^-]$ subject to the single global relation $\sum_i([A_i^+]+[A_i^-]) = 0$. We will slightly abuse notation and suppress the square brackets from such formulas below.

LEMMA 6.3. *Any single element $\gamma_1 \in T$ can be realized by a simple closed curve in $\widehat{F}$.*

Proof. Write $\gamma_1 = \sum_j \varepsilon_{1j}\alpha_j$ as above, with $\varepsilon_{1j} \in \{0, \pm 1\}$. By replacing some of the α_j with $-\alpha_j$ as necessary, we can assume that $\gamma_1 = \sum_{j=1}^k \alpha_j$. The corresponding homology class $\widehat{\gamma_1} = \sum_{j=1}^k A_j^+$ in $\widehat{F}$ is then represented by a simple closed curve, as required.

LEMMA 6.4. *If α_j and α_k both have nonzero coefficients in the expansions of both of γ_1 and γ_2, then either $\varepsilon_{1j} = \varepsilon_{2j}$ and $\varepsilon_{1k} = \varepsilon_{2k}$ or $\varepsilon_{1j} = -\varepsilon_{2vj}$ and $\varepsilon_{1k} = -\varepsilon_{2k}$. That is, the coefficients either agree or disagree.*

Proof. If not, the matrix M representing the γ_i has a 2 by 2 submatrix of the form

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

up to multiplying rows and/or columns by -1, contradicting the matrix interpretation of the Summand Condition as given in Lemma 6.2.

For $\gamma_i \in T$ define its *support* (with respect to $\{\alpha_1, \dots, \alpha_n\}$) to be the set of α_j with nonzero coefficient in the expression $\gamma_i = \sum_j \varepsilon_{ij}\alpha_j$. Note that up to relabeling there are just three ways for the supports of γ_1 and γ_2 in T to relate to one another. The two classes may have nested supports, disjoint supports, or properly overlapping supports.

LEMMA 6.5. *Any two distinct elements $\gamma_1, \gamma_2 \in T$ can be realized by disjoint simple closed curves in $\widehat{F}$.*

Proof. There are three cases to consider, organized by the relative placement of the supports. Without loss of generality we can assume that card supp $\gamma_1 \geq$ card supp γ_2. Then either

(1) supp $\gamma_2 \subset$ supp γ_1 or

(2) supp $\gamma_1 \cap$ supp $\gamma_2 = \varnothing$ or

(3) supp $\gamma_1 \cap$ supp $\gamma_2 \neq \varnothing$ and supp$\gamma_1 \cap$ supp $\gamma_2 \neq$ supp γ_2.

As in the proof of Lemma 6.3 we may assume that $\gamma_1 = \sum_{j=1}^k \alpha_j$.

In case (1) we may, by Lemma 6.4, assume that γ_2 has the form $\sum_{j=1}^{\ell} \alpha_j$ for some $\ell < k$. Then the two corresponding classes $\widehat{\gamma}_1 = \sum_{j=1}^{k} A_j^+$ and $\widehat{\gamma}_2 = \sum_{j=1}^{\ell} A_j^+$ can be realized disjointly in $\widehat{F}$ as required, by Proposition 5.2.

In case (2) we may assume that γ_2 has the form $\sum_{j=k+1}^{\ell} \alpha_j$ for some $\ell > k$. Then the two corresponding classes $\widehat{\gamma}_1 = \sum_{j=1}^{k} A_j^+$ and $\widehat{\gamma}_2 = \sum_{j=k+1}^{\ell} A_j^+$ can be realized disjointly in $\widehat{F}$ as required.

In case (3) we may assume, again by Lemma 6.4, that γ_2 has the form $\sum_{j=r}^{s} \alpha_j$ for some $r \leq k$ and $s > k$. Then the two corresponding classes $\widehat{\gamma}_1 = \sum_{j=1}^{k} A_j^+$ and $\widehat{\gamma}_2 = \sum_{j=r}^{s} A_j^-$ can be realized disjointly in $\widehat{F}$ as required.

PROPOSITION 6.6. *Any three distinct elements γ_1, γ_2, γ_3 in T can be realized by disjoint simple closed curves in $\widehat{F}$.*

Proof. Once again we organize the analysis according to the relative positions of the supports of the three homology classes. There are several cases to consider. In each of several cases we shall normalize the expressions for the γ_i in terms of the α_j. The operations we will use are permutation of the γ_i, permutation of the α_j, changing the sign of one or more γ_i, and changing the sign of one or more α_j.

Suppose that the support of one class is contained in the support of another class. Without loss of generality we may assume that

$$\gamma_1 = \sum_{j=1}^{k} \alpha_j \quad \text{and} \quad \gamma_2 = \sum_{j=1}^{\ell} \alpha_j \quad \text{for some} \quad \ell < k .$$

Now the signs of all coefficients of γ_3 involved in γ_1 may be assumed to be negative, by Lemma 6.4. So we may assume that

$$\gamma_3 = \sum_{j=u}^{v} -\alpha_j$$

Then the three preferred lifts

$$\widehat{\gamma}_1 = \sum_{j=1}^{k} A_j^+ , \quad \widehat{\gamma}_2 = \sum_{j=1}^{\ell} A_j^+ , \quad \text{and} \quad \widehat{\gamma}_3 = \sum_{j=u}^{v} A_j^-$$

can clearly be realized disjointly in $\widehat{F}$. Henceforth we may assume that no one of the three given homology classes has its support contained in the support of one of the others.

If the underlying support of one of the 3 classes, say of γ_3, is disjoint from the supports of both of the other two, then this is easy. Realize $\widehat{\gamma_1}$ and $\widehat{\gamma_2}$ as above; then realize the preferred lift $\widehat{\gamma_3}$ of γ_3, which has support disjoint from those of $\widehat{\gamma_1}$ and $\widehat{\gamma_2}$.

Suppose now that two classes have disjoint support, but that no homology class has support disjoint from the supports of both of the other two. Without loss of generality we may assume that

$$\gamma_1 = \sum_{j=1}^{k} \alpha_j \quad \text{and} \quad \gamma_2 = \sum_{j=k+1}^{\ell} \alpha_j$$

for some $\ell > k+1$. Now γ_3 involves some, but not all, of the support of γ_1, some, but not all, of the support of γ_2, and, perhaps, some terms not involved in either of γ_1 or γ_2. After permuting basis elements we have $\gamma_3 = \sum_{j=r}^{s} \varepsilon_{3j}\alpha_j + \sum_{j=\ell+1}^{n} \varepsilon_{3j}\alpha_j$, where $2 \le r \le k-1$, $k+1 \le s \le \ell - 1$. Now the Summand Condition implies that all ε_{3j}, $r \le j \le k$, have the same sign; and all ε_{3j}, $k+1 \le j \le s$, have the same sign. By changing the global sign of γ_3 if necessary we may assume that $\varepsilon_{3j} = -1$ for $r \le j \le k$. Further, by changing the sign of α_i, $i > \ell$ as needed we may assume that $\varepsilon_{3j} \le 0$ for $i > \ell$. Thus at this point we have arranged that

$$\gamma_3 = -\sum_{j=r}^{k} \alpha_j \pm \sum_{j=k+1}^{s} \alpha_j - \sum_{j=\ell+1}^{t} \alpha_j$$

where $2 \le r \le k-1$, $k+1 \le s \le \ell - 1$, and $\ell + 1 \le t \le n$, and the third sum might not actually appear at all. If the "$-$" sign prevails in the formula for γ_3, then the preferred lifts of γ_1, γ_2, and γ_3 are disjointly realizable in $\widehat{F}$ as required. On the other hand, if the "+" sign prevails we can reduce to the previous case by first replacing $\alpha_{k+1}, \dots, \alpha_\ell$ with their negatives, and then replacing γ_2 with its negative.

Now we may suppose for the rest of the argument that no two classes have disjoint support, and that no class has support contained in the support of one of the other two classes.

For the penultimate case suppose that the intersection of all three supports is empty. We divide the supports of the γ_i three pieces : $S_{ij} = \operatorname{supp}\gamma_i \cap \operatorname{supp}\gamma_i$ and $T_i = \operatorname{supp}\gamma_i - \operatorname{supp}\gamma_j \cup \operatorname{supp}\gamma_k$, where $\{i,j,k\} = \{1,2,3\}$. In what follows we will, for notational simplicity, sometimes identify α_j with its index j. Then, without loss of generality, after changing the signs of various α_i as necessary, we can assume that

$$\gamma_1 = \sum_{i\in S_{12}} \alpha_i + \sum_{i\in S_{13}} \alpha_i + \sum_{i\in T_1} \alpha_i .$$

Then, replacing γ_2 by its negative if necessary, and changing the sign of α_i, $i \in S_{23} \cup T_2$ as necessary, and invoking the 2×2 Summand Condition, we can assume that

$$\gamma_2 = -\sum_{i \in S_{12}} \alpha_i - \sum_{i \in S_{23}} \alpha_i - \sum_{i \in T_2} \alpha_i .$$

Similarly, we can arrange that

$$\gamma_3 = -\sum_{i \in S_{13}} \alpha_i \pm \sum_{i \in S_{23}} \alpha_i + \sum_{i \in T_3} \alpha_i .$$

Now the 3×3 Summand Condition tells us that the $+$ sign must prevail in the expansion of γ_3. For otherwise the matrix M would contain a 3×3 submatrix of the form

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

which has determinant -2. Now with all these normalizations, the preferred lifts of γ_1, γ_2, and γ_3 are disjointly realizable in $\widehat{F}$ as required.

Finally, at last, we have the case that the intersection of all three supports is nonempty but that no support set is contained in one of the other supports. Let $S_i = \operatorname{supp} \gamma_i$, $S_{ij} = S_i \cap S_j$, and $S_{123} = S_1 \cap S_2 \cap S_3 \neq \varnothing$. Now as always we can assume one of our homology classes, say γ_1 has all nonnegative coefficients, i.e., $\gamma_1 = \sum_{j \in S_1} \alpha_j$. Next we can assume by the 2 by 2 Summand Condition that γ_2 has positive coefficients on S_{12}, and of course that it has positive coefficients on $S_2 - S_{12}$. In particular, then, we have $\gamma_2 = \sum_{j \in S_2} \alpha_j$. Since $S_{123} \neq \varnothing$, all coefficients of elements of $S_3 \cap (S_1 \cup S_2)$ must have the same sign, which we may assume is positive. It follows that we may arrange that $\gamma_3 = \sum_{j \in S_3} \alpha_j$. In this case the preferred lifts of the γ_i will not be disjointly realizable and we have to choose other lifts as follows. For γ_1 we do use the preferred lift

$$\widehat{\gamma}_1 = \sum_{j \in S_1} A_j^+ .$$

For γ_2, however, we add on to the preferred lift canceling pairs corresponding to elements of $S_1 - S_2$ and define

$$\widehat{\gamma}_2 = \sum_{j \in S_2} A_j^+ + \sum_{j \in S_1 - S_2} (A_j^+ + A_j^-)$$

and, finally, for γ_3 we define

$$\widehat{\gamma}_3 = \sum_{j \in S_3} A_j^+ + \sum_{j \in S_1 \cup S_2 - S_3} (A_j^+ + A_j^-) .$$

These choices of lifts of the γ_i to homology classes in $\widehat{F}$ satisfy the conditions for disjoint realizability given in Section 5. (We emphasize again that the conditions for realizability in planar surfaces continue to hold for homology classes in nonplanar compact surfaces with boundary provided the homology classes in question are all linear combinations of the classes of the boundary curves.)

The one remaining aspect to consider in the proof of the sufficiency part of the Theorem 3 is given by the following result.

PROPOSITION 6.7. *If* rank $S \leq 4$, *then* S *can be realized by disjoint simple closed curves in* $\widehat{F}$.

Proof sketch. We will only outline the proof, which is a tedious case-by-case check. In light of the preceding results, we may assume that F has genus 4 and that S consists of $\alpha_1, \ldots, \alpha_4$ together with 4 or 5 additional classes γ_i. We describe the system of γ_i by a matrix with 4 or 5 rows and 4 columns, consisting of entries 1, -1, or 0. (Conversely, any such matrix determines a collection of homology classes which one can try to realize.) We can normalize each such matrix according to the following principles: First of all we can arrange that the rows have monotonically nonincreasing size of support as one goes down the rows. Next, within any collection of rows with supports of the same size we can assume that the rows appear in lexicographical ordering according to the alphabet ordering $+1$, -1, 0. Next, by changing signs of the elements of S as required we can assume that the first nonzero element in each row and in each column is $+1$. A computer can easily crank out a list of all such matrices in lexicographical order. (It follows from the Summand Condition that there is at most one element of length 4 (i.e., involving all 4 classes α_i). Similarly, there are at most 2 elements of length 3. Again this follows from the Summand Condition, since two elements of length 3 must have two support elements in common and since the coefficients of the α_i appearing in the overlap of the supports of two classes must all have the same signs. The remaining classes must have support size 2.) At this point one should check the Summand Condition by checking that the determinant of every square submatrix is also $+1$, -1, or 0 and throw out those that do not meet this condition. Finally, in any particular case there may be extra symmetries at hand, exchanging pairs of rows or pairs of columns so as to produce a matrix higher up on our list. This last step is done by hand. Ultimately in this way we produce a list of 36 such matrices which one must show are realizable by actually drawing an appropriate planar diagram as above. (There is some redundancy in that some of the 4 by 4 matrices appear as submatrices of 5 by 4 matrices later in the list.) Although some of the required diagrams were a little difficult to discover, in the end all 36 were shown to be realizable. As just one example, here is one of the trickier realizable families:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

This corresponds to the family

$$S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_4, \alpha_2 - \alpha_3, \alpha_2 - \alpha_4\}$$

The following classes on the 4×2-punctured sphere lift the five composite classes:

$$A_1^+ + A_2^+ + A_3^+ + A_3^-, A_1^+ + A_3^+, A_1^+ + A_4^+, A_2^+ + A_3^-, A_2^- + A_4^+$$

This collection of classes can be realized by pairwise disjoint simple closed curves on the punctured sphere, and this realization then descends to give a realization of the given homology classes.

Discussion of the proof of Theorem 6, an algorithmic solution to the realizability problem. The results of Section 5 on realizing curves on a planar surface and of the first part of this Section 6, combine to provide a finite algorithm for realizing any family of homology classes by pairwise disjoint simple closed curves. As usual, let $S \subset H_1(F)$ denote a finite set of distinct homology classes satisfying the Intersection Condition, the Summand Condition, and the Size Condition of the the Main Theorem, which we wish to represent by pairwise disjoint simple closed curves. Suppose that the linear span of S has rank n and extract from S n elements $\alpha_1, \ldots, \alpha_n$ that form a basis for this span. Now each element γ_i in the remaining part of S can be expressed as a linear combination

$$\gamma_i = \sum_j \varepsilon_{ij}\, \alpha_j$$

in which we know by Lemma 6.1 that the coefficients satisfy $|\varepsilon_{ij}| \leq 1$. Moreover, we may also assume that we have already represented elements $\alpha_1, \ldots, \alpha_n$ by disjoint simple closed curves $A_1, \ldots, A_n$, by Proposition 4.2.

We attempt to represent the remaining classes in T, the complement of $\alpha_1,\dots,\alpha_n$ in S. Let $\widehat{F}$ denote F cut open along the A_i. Then $\widehat{F}$ is a 2-sphere with $2n$ holes (possibly with some additional handles attached, which play no role in the present discussion and which can safely be ignored), with orientable boundary consisting of one copy A_i^+ of each A_i and one copy A_i^- of each A_i with its orientation reversed. As in Section 5, the relevent homology $\mathcal{B} \subset H_1(\widehat{F})$ is generated by the homology classes of the boundary curves. Now the set S of homology classes can be realized by pairwise disjoint simple closed curves in F if and only if the classes in T can be lifted to a set $\widehat{T}$ of homology classes in $\mathcal{B} \subset H_1(\widehat{F})$ that can be realized by pairwise disjoint simple closed curves in $\widehat{F}$. Now, the classes γ_i have infinitely many pre-images in $H_1(\widehat{F})$, but only finitely many pre-images can be represented by simple closed curves, since by Lemma 5.1 the coefficients of the classes of the boundary curves must have absolute value at most 1, and all must have the same sign. There are only finitely many such lifts of each homology class and they may all be considered one-by-one for realizability using Proposition 5.3, which is itself finitely verifiable.

7. Various instructive examples

Here we present three interesting examples that point to some of the difficulties in finding necessary *and sufficient* conditions for realizability of a system of homology classes by pairwise disjoint simple closed curves. To start with we give an example showing that even when a system is realizable it is possible to get stuck, in the sense that one might realize all but one class and then have no way to realize the last class so as to be disjoint from the other curves. In particular, one might have to go back and change the curves already realized in order to complete the construction.

Example 7.1. Non-extendable partial realizations of a realizable family of homology classes.

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1+\alpha_2, \alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4\}$ be a system of homology classes on a surface of genus 4, in which $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is part of a standard symplectic basis. One can check that this collection satisfies all the necessary conditions in the Theorem 1. By Theorem 3, it is realizable by a system of pairwise disjoint simple closed curves. Explicitly, we can first realize $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ by standard curves A_1, A_2, A_3, A_4 in the