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THE ZERO-IN-THE-SPECTRUM QUESTION

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ABSTRACT. This is an expository article on the question of whether zero lies in the spectrum of the Laplace-Beltrami operator acting on differential forms on a manifold.

1. INTRODUCTION

Let M be a complete connected oriented Riemannian manifold. The Laplace-Beltrami operator $\triangle_p$ acts on the square-integrable p-forms on M. We asked the following question in 1991 :

ZERO-IN-THE-SPECTRUM QUESTION. Is zero always in the spectrum of $\triangle_p$ for some p ?

To our knowledge, nobody has found a counterexample. The question was also raised by Gromov in the case of a contractible manifold with a discrete cocompact group of isometries ([15], p. 21).

Being able to answer the above question is a first step toward understanding the spectrum of the Laplace-Beltrami operator. We would also like to be able to say whether or not zero is in the spectrum of $\triangle_p$ for a given p. This problem is partly topological in nature and partly geometric, in a sense which will be made precise later. In fact, it is equivalent to knowing the (unreduced) L^2-cohomology of M. The study of L^2-cohomology touches on many branches of mathematics, including combinatorial group theory, topology, differential geometry and algebraic geometry. It is most commonly considered when M is the universal cover of a compact manifold or when M is a finite-volume

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Hermitian locally symmetric space. We refer to [22, 26] and [30] for surveys of these two cases. In this article we will instead emphasize general complete Riemannian manifolds and give some partial positive answers to the zero-in-the-spectrum question.

The sections of the article are:

In what follows, all manifolds will be smooth, connected, oriented and of positive dimension. All maps between manifolds will be orientation-preserving. Unless otherwise indicated, all Riemannian manifolds will be complete.

We have tried to give as many complete proofs as reasonably possible. All unattributed results are of unknown origin or are due to the author. I thank Wolfgang Lück for conversations on some of the topics discussed herein. I thank Marie-Claude Vergne for making the figures. This article is based on lectures given at the Troisième Cycle Romand "On the Conjecture of the Zero in the Spectrum" held at Les Diablerets, Switzerland, March, 1996. I warmly thank Alain Valette and the other organizers and participants of the meeting.