DEFINITION 3. *We say that* $\mathrm{H}^j(M;\mathbf{C})$ *vanishes uniformly if for all* $r>0$, *there is an* $R(r)\geq r$ *such that for all* $m\in M$,

$$\operatorname{Im}\left(H^j\left(B_{R(r)}(m);\mathbf{C}\right)\rightarrow H^j\left(B_r(m);\mathbf{C}\right)\right)=0. \tag{2.11}$$

PROPOSITION 3 (Pansu [25]). *Consider a Riemannian manifold* M *of bounded geometry such that for some* $k>0$, $\mathrm{H}^j(M;\mathbf{C})$ *vanishes uniformly for* $1\leq j\leq k$. *Then within the class of such manifolds,*

1. $\overline{\mathrm{H}}^p_{(2)}(M)$ *and* $\mathrm{H}^p_{(2)}(M)$ *are coarse quasi-isometry invariants for* $0\leq p\leq k$.
2. $\operatorname{Ker}\left(\overline{\mathrm{H}}^{k+1}_{(2)}(M)\rightarrow \mathrm{H}^{k+1}(M;\mathbf{C})\right)$ *and* $\operatorname{Ker}\left(\mathrm{H}^{k+1}_{(2)}(M)\rightarrow \mathrm{H}^{k+1}(M;\mathbf{C})\right)$ *are coarse quasi-isometry invariants.*

3. GENERAL PROPERTIES OF L^2-COHOMOLOGY

In this section we give some general results about the L^2-cohomology of complete Riemannian manifolds. First, we give a useful sufficient condition for the reduced L^2-cohomology to be nonzero.

PROPOSITION 4. *For all* p, $\operatorname{Im}\left(\mathrm{H}^p_c(M;\mathbf{C})\rightarrow \mathrm{H}^p(M;\mathbf{C})\right)$ *injects into* $\overline{\mathrm{H}}^p_{(2)}(M)$.

Proof. Suppose that ω is a smooth compactly-supported closed p-form which represents a nonzero class in $\mathrm{H}^p(M;\mathbf{C})$. By Poincaré duality, there is a smooth compactly-supported closed $\left(\dim(M)-p\right)$-form ρ such that $\int_M \omega\wedge\rho\neq 0$.

As ω is compactly-supported, it is square-integrable and so represents an element $[\omega]$ of $\overline{\mathrm{H}}^p_{(2)}(M)$. Suppose that $[\omega]=0$. Then there is a sequence $\{\eta_i\}_{i=1}^{\infty}$ in $\Omega^{p-1}(M)$ such that $\omega=\lim_{i\to\infty} d\eta_i$, where the limit is in an L^2-sense. It follows that

$$\int_M \omega\wedge\rho=\lim_{i\to\infty}\int_M d\eta_i\wedge\rho=\lim_{i\to\infty}\int_M d(\eta_i\wedge\rho)=0, \tag{3.1}$$

which is a contradiction. Thus $[\omega]\neq 0$. □

COROLLARY 2. *Let N^{4k} be a compact manifold-with-boundary with nonzero signature. Then if M is any complete Riemannian manifold which is diffeomorphic to* $\mathrm{int}(N)$, $\overline{\mathrm{H}}^{2k}_{(2)}(M) \neq 0$.

Proof. By definition, the signature of N is the signature of the intersection form on

$$\text{(3.2)} \quad \mathrm{Im}\left(\mathrm{H}^{2k}(N, \partial N; \mathbf{C}) \to \mathrm{H}^{2k}(N; \mathbf{C})\right) \cong \mathrm{Im}\left(\mathrm{H}^{2k}_c(M; \mathbf{C}) \to \mathrm{H}^{2k}(M; \mathbf{C})\right).$$

If the signature of N is nonzero then $\mathrm{Im}\left(\mathrm{H}^{2k}_c(M; \mathbf{C}) \to \mathrm{H}^{2k}(M; \mathbf{C})\right)$ must be nonzero. The corollary follows from Proposition 4. □

EXAMPLE. Let N be $\mathbf{C}P^2$ with a small 4-ball removed. Then N satisfies the hypothesis of Corollary 2.

We now show that the middle-dimensional reduced L^2-cohomology is a conformal invariant of M.

PROPOSITION 5. *If M^{2k} is even-dimensional then* $\mathrm{Ker}(\triangle_k)$ *is conformally-invariant.*

Proof. Suppose that g and $e^{\phi}g$ are conformally equivalent Riemannian metrics on M, with $\phi \in C^\infty(M)$. We use the fact that the action of the Hodge duality operator $*$ on $\Lambda^k(M)$ is independent of ϕ. If ω is a k-form on M, its L^2-norm $\int_M \omega \wedge *\omega$ is independent of ϕ. Thus the Hilbert space $\Lambda^k(M)$ is independent of ϕ. Furthermore,

$$\begin{aligned}
\text{(3.3)} \qquad \mathrm{Ker}(\triangle_k) &= \{\omega \in \Lambda^k(M) : d\omega = d^*\omega = 0\} \\
&= \{\omega \in \Lambda^k(M) : d\omega = \pm * d * (\omega) = 0\} \\
\text{(3.4)} \qquad &= \{\omega \in \Lambda^k(M) : d\omega = d * (\omega) = 0\}
\end{aligned}$$

is independent of ϕ. □

EXAMPLE. Take $M = H^2$. Then M is conformally equivalent to a Euclidean disk D. The harmonic square-integrable 1-forms on D are of the form $f_1(x,y)\,dx + f_2(x,y)\,dy$, where f_1 and f_2 are square-integrable harmonic functions on D. There is clearly an infinite number of such functions, and so $\dim\left(\overline{\mathrm{H}}^1_{(2)}(H^2)\right) = \infty$. The same argument applies to H^{2k}, to give $\dim\left(\overline{\mathrm{H}}^k_{(2)}(H^{2k})\right) = \infty$.

In the case of functions, one has a good control of when zero is in the spectrum of the Laplacian.

LEMMA 4. $\operatorname{Ker}(\triangle_0) \neq 0$ *if and only if* $\operatorname{vol}(M) < \infty$.

Proof. If $\operatorname{vol}(M) < \infty$ then the constant functions on M are square-integrable and harmonic. Conversely, if $f \in \operatorname{Ker}(\triangle_0)$ then by Lemma 2, f is constant. If f is nonzero and square-integrable then $\operatorname{vol}(M) < \infty$.

DEFINITION 4. *M is open at infinity if there is a constant $C > 0$ such that for all domains D in M with smooth compact closure,* $\frac{\operatorname{area}(\partial D)}{\operatorname{vol}(D)} \geq C$.

EXAMPLES.

1. $\mathbf{R}^n$ is not open at infinity, as can be seen by taking large balls for D.
2. H^n is open at infinity.

PROPOSITION 6 (Buser [3]). *Let M have infinite volume. Suppose that there is a constant $c \geq 0$ such that* $\operatorname{Ricci}_M \geq -c^2$. *Then $0 \notin \sigma(\triangle_0)$ if and only if M is open at infinity.*

Proof.

1. Suppose that M is open at infinity. By Cheeger's inequality,

(3.5) $$\inf\bigl(\sigma(\triangle_0)\bigr) \geq \inf_D \frac{1}{4}\left(\frac{\operatorname{area}(\partial D)}{\operatorname{vol}(D)}\right)^2 > 0.$$

2. Suppose that M is not open at infinity. The bottom of the spectrum of $\triangle_0$ is given in terms of Rayleigh quotients by

(3.6) $$\inf\bigl(\sigma(\triangle_0)\bigr) = \inf_{f\neq 0} \frac{\int_M |df|^2}{\int_M f^2},$$

where f ranges over compactly-supported Lipschitz functions on M. We want to find compactly-supported Lipschitz functions on M of arbitrarily small Rayleigh quotient. By assumption, for all $\epsilon > 0$ there is a domain D such that $\frac{\operatorname{area}(\partial D)}{\operatorname{vol}(D)} \leq \epsilon$. Put

(3.7) $$N_1(\partial D) = \{m \in M : m \notin D \text{ and } d(m, \partial D) \leq 1\}.$$

Define a function f, which approximates the characteristic function of D, by

(3.8) $$f(m) = \begin{cases} 1 & \text{if } m \in D \\ 1 - d(m, \partial D) & \text{if } m \in N_1(\partial D) \\ 0 & \text{if } m \notin D \text{ and } m \notin N_1(\partial D). \end{cases}$$

Clearly $\int_M f^2 \geq \mathrm{vol}(D)$. As f has nonzero gradient only in $N_1(\partial D)$, where $|df| = 1$ almost everywhere, we have $\int_M |df|^2 = \mathrm{vol}\big(N_1(\partial D)\big)$. If D is nice and round then we expect that

$$\mathrm{vol}\big(N_1(\partial D)\big) \sim \mathrm{area}(\partial D) \tag{3.9}$$

and the Rayleigh quotient $\dfrac{\int_M |df|^2}{\int_M f^2}$ will be comparable to ϵ.

The only problem with this argument is that D may not be nice and round, but may have long thin legs coming out of it, like an octopus. Then (3.9) may not be valid. The content of [3] is that if this is the case, we can cut the legs off of D to come up with a new domain for which the above heuristic argument is valid. We refer to [3] for details. □

COROLLARY 3 (Brooks [2]). *Let M be a normal covering of a compact manifold X with covering group Γ. Then $0 \in \sigma(\triangle_0)$ on M if and only if Γ is amenable.*

Proof. If Γ is finite then $0 \in \sigma(\triangle_0)$ and Γ is amenable. If Γ is infinite then by Proposition 6, $0 \in \sigma(\triangle_0)$ if and only if M is not open at infinity. Let S be a finite set of generators of Γ. Let G be the Cayley graph of Γ, constructed using S. There is a notion of G being open at infinity which is similar to Definition 4. As M is coarsely quasi-isometric to G, M is not open at infinity if and only if G is not open at infinity. However, this is one of the characterizations of amenability of Γ. □

We now prove a result about manifolds which, roughly speaking, are at least as large as Euclidean space.

DEFINITION 5. *M is hyperEuclidean if there is a proper distance-nonincreasing map $F : M \to \mathbf{R}^{\dim(M)}$ of nonzero degree.*

REMARKS.

1. A map is proper if preimages of compact sets are compact. Instead of requiring that F be distance-nonincreasing, we could require that F have a finite Lipschitz constant. By postcomposing F with a dilatation of $\mathbf{R}^{\dim(M)}$, the two conditions are equivalent.
2. If M is hyperEuclidean then a compactly-supported modification of M is also hyperEuclidean.

3. Examples of hyperEuclidean manifolds are given by simply-connected nonpositively-curved manifolds M. Namely, fix $m_0 \in M$ and put $F = \exp_{m_0}^{-1}$.
4. There was once a conjecture that all uniformly contractible manifolds are hyperEuclidean (with a degree-one map to $\mathbf{R}^{\dim(M)}$), but this turns out to be wrong [11]. There is still an open conjecture that a uniformly contractible manifold of bounded geometry is hyperEuclidean, and in particular, that the universal cover of an aspherical closed manifold is hyperEuclidean.

PROPOSITION 7 (Gromov [15, p. 238]). *If M is hyperEuclidean then $0 \in \sigma(\triangle_p)$ for some p.*

Proof. Put $n = \dim(M)$. First, suppose that n is even. We will construct a vector bundle E with connection on $\mathbf{R}^n$ which is topologically nontrivial but analytically trivial, in a sense which will be made precise. Then assuming that zero is not in the spectrum of M, we will apply the relative index theorem to F^*E in order to get a contradiction.

Recall that $\mathrm{K}^0(S^n) = \mathbf{Z} \oplus \mathbf{Z}$. If $\mathcal{E}$ is a (virtual) vector bundle on S^n, the two $\mathbf{Z}$ factors correspond to $\mathrm{rk}(\mathcal{E})$ and $\int_{S^n} \mathrm{ch}(\mathcal{E})$, respectively. This means that for some $N > 0$, there is a $\mathbf{C}^N$-bundle $\mathcal{E}$ on S^n with $\int_{S^n} \mathrm{ch}(\mathcal{E}) \neq 0$. Fixing a point $\infty \in S^n$, we can trivialize $\mathcal{E}$ in a neighborhood of ∞. Furthermore, we can put a Hermitian metric and Hermitian connection on $\mathcal{E}$ so that the connection is flat in a neighborhood of ∞.

Let E be the restriction of $\mathcal{E}$ to $\mathbf{R}^n = S^n - \{\infty\}$. Let ∇ be the restriction of the Hermitian connection on $\mathcal{E}$ to $\mathbf{R}^n$. Then E is trivialized outside of a compact set $K \subset \mathbf{R}^n$ and ∇ is flat outside of K.

As $\mathbf{R}^n$ is contractible, there is an isomorphism of Hermitian vector bundles $i : \mathbf{R}^n \times \mathbf{C}^N \to E$. Then $i^*\nabla$ can be considered to be a $u(N)$-valued 1-form ω on $\mathbf{R}^n$. The curvature of ω is the $u(N)$-valued 2-form $\Omega = d\omega + \omega^2$. The nontriviality of $\mathcal{E}$ translates to the facts that

1. Ω vanishes outside of K and
2. The de Rham cohomology class of the compactly-supported form

$$\mathrm{Tr}\left(e^{-\frac{\Omega}{2\pi i}}\right) - N$$

is a nonzero multiple of the fundamental class $[\mathbf{R}^n] \in \mathrm{H}_c^n(\mathbf{R}^n; \mathbf{R})$.

In fact, we can take ω to have a finite L^∞-norm $\| \omega \|_\infty$. For example, if $n = 2$, take $N = 1$. Let $f \in C_0^\infty([0, \infty))$ be a nonincreasing function such that if $x \in [0, 1]$ then $f(x) = 1$. Put $\omega = -i\big(1 - f(r)\big)\, d\theta$. Then

$$\Omega = d\omega = i f'(r) dr \wedge d\theta\,. \tag{3.10}$$

We have $\| \omega \|_\infty = \sup_{r>0} \frac{1-f(r)}{r}$ and $\int_{\mathbf{R}^2} \left[\mathrm{Tr}\left(e^{-\frac{\Omega}{2\pi i}} \right) - 1 \right] = 1$.

Returning to the case of general even n, for $\epsilon > 0$, let $\Phi_\epsilon : \mathbf{R}^n \to \mathbf{R}^n$ be the map $\Phi_\epsilon(\mathbf{x}) = \epsilon \mathbf{x}$. Put $\omega_\epsilon = \Phi_\epsilon^* \omega$ and $\Omega_\epsilon = d\omega_\epsilon + \omega_\epsilon^2$. Then

$$\begin{aligned} \| \omega_\epsilon \|_\infty = \epsilon \| \omega \|_\infty \quad \text{and} \quad &\int_{\mathbf{R}^n} \left[\mathrm{Tr}\left(e^{-\frac{\Omega_\epsilon}{2\pi i}} \right) - N \right] \\ &= \int_{\mathbf{R}^n} \left[\mathrm{Tr}\left(e^{-\frac{\Omega}{2\pi i}} \right) - N \right] \neq 0\,. \end{aligned} \tag{3.11}$$

We now turn our attention to M. Suppose that $0 \notin \sigma(\triangle_p)$ for all p. Consider the self-adjoint operator $d + d^*$ on $\Lambda^*(M)$. As $(d+d^*)^2 = \triangle$, it follows that $0 \notin \sigma(d + d^*)$. In other words, $d + d^*$ is L^2-invertible. Define an operator μ on $\Lambda^*(M)$ by saying that if $\omega \in \Lambda^p(M)$ then

$$\mu(\omega) = i^{\frac{n(n-1)}{2}} (-1)^{\frac{p(p-1)}{2}} * (\omega)\,. \tag{3.12}$$

One can check that $\mu^2 = 1$ and $\mu(d + d^*) + (d + d^*)\mu = 0$.

Clearly the operator $(d+d^*) \otimes \mathrm{Id}_N$, acting on $\Lambda^*(M) \otimes \mathbf{C}^N$, is also invertible. Consider the $u(N)$-valued 1-form $F^*\omega_\epsilon$ on M. As F is distance-nonincreasing,

$$\| F^*\omega_\epsilon \|_\infty \leq \| \omega_\epsilon \|_\infty = \epsilon \| \omega \|_\infty\,. \tag{3.13}$$

Let $e(F^*\omega_\epsilon)$ denote exterior multiplication by $F^*\omega_\epsilon$, acting on $\Lambda^*(M) \otimes \mathbf{C}^N$ and let $i(F^*\omega_\epsilon)$ denote interior multiplication by $F^*\omega_\epsilon$. By making ϵ small enough, the operator $e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon)$ has arbitrarily small norm and so the operator $\big((d + d^*) \otimes \mathrm{Id}_N\big) + e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon)$ is also invertible.

Put $D = (d \otimes \mathrm{Id}_N) + e(F^*\omega_\epsilon)$. Then D is exterior differentiation, using the connection $F^*\omega_\epsilon$, and

$$D + D^* = \big((d + d^*) \otimes \mathrm{Id}_N\big) + e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon)\,. \tag{3.14}$$

As $(d + d^*) \otimes \mathrm{Id}_N$ and $D + D^*$ anticommute with $\mu \otimes \mathrm{Id}_N$, they have well-defined indices which happen to vanish, as the operators are invertible. On the other hand, let $L(M)$ be the Hirzebruch L-form. The relative index theorem of Gromov and Lawson [10, 16] says that

$$\begin{aligned} &\mathrm{ind}(D + D^*) - \mathrm{ind}\big((d + d^*) \otimes \mathrm{Id}_N\big) \\ &\qquad = \int_M L(M) \wedge \left[\mathrm{Tr}\left(e^{-\frac{F^*\Omega_\epsilon}{2\pi i}} \right) - N \right]. \end{aligned} \tag{3.15}$$

As F is proper, the de Rham cohomology class of $\mathrm{Tr}\left(e^{-\frac{F^*\Omega_\epsilon}{2\pi i}}\right) - N = F^*\left[\mathrm{Tr}\left(e^{-\frac{\Omega_\epsilon}{2\pi i}}\right) - N\right]$ is well-defined as a multiple of the fundamental class $[M] \in \mathrm{H}^n_c(M;\mathbf{R})$. As the series for $L(M)$ starts off as $L(M) = 1 + \dots$, we obtain

$$\begin{aligned} \mathrm{ind}(D+D^*) - \mathrm{ind}\big((d+d^*)\otimes \mathrm{Id}_N\big) &= \int_M \left[\mathrm{Tr}\left(e^{-\frac{F^*\Omega_\epsilon}{2\pi i}}\right) - N\right] \\ &= \int_M F^*\left[\mathrm{Tr}\left(e^{-\frac{\Omega_\epsilon}{2\pi i}}\right) - N\right] \\ &= \deg(F)\int_{\mathbf{R}^n}\left[\mathrm{Tr}\left(e^{-\frac{\Omega_\epsilon}{2\pi i}}\right) - N\right] \neq 0. \end{aligned} \tag{3.16}$$

This contradicts the vanishing of $\mathrm{ind}(D+D^*)$ and $\mathrm{ind}\big((d+d^*)\otimes \mathrm{Id}_N\big)$. Thus zero must be in the spectrum of M after all.

Now suppose that n is odd. As M is hyperEuclidean, so is $\mathbf{R}\times M$. With respect to the decomposition $\Lambda^*(\mathbf{R}\times M) = \Lambda^*(\mathbf{R})\otimes\Lambda^*(M)$, the Laplace-Beltrami operator on $\mathbf{R}\times M$ decomposes as

$$\triangle_{\mathbf{R}\times M} = (\triangle_{\mathbf{R}}\otimes I) + (I\otimes\triangle_M). \tag{3.17}$$

Then

$$\sigma(\triangle_{\mathbf{R}\times M}) = \{\lambda_1+\lambda_2 : \lambda_1\in[0,\infty) \quad \text{and} \quad \lambda_2\in\sigma(\triangle_M)\}. \tag{3.18}$$

From what has already been proved, $0 \in \sigma(\triangle_{\mathbf{R}\times M})$. It follows that $0\in\sigma(\triangle_M)$. □

REMARKS.

1. We have shown that if M is hyperEuclidean then $0\in\sigma(\triangle_p)$ for some p. One can ask whether the number p can be pinned down. In general, when computing the index of the operator $d+d^*$, the differential forms outside of the middle dimensions do not contribute. This is a reflection of the fact that the signature of a closed manifold can be computed using only the middle-dimensional cohomology. So this gives some reason to think that if $\dim(M)$ is even then $0\in\sigma\left(\triangle_{\frac{\dim(M)}{2}}\right)$.

Unfortunately, the operator $(D+D^*)^2$ does not preserve the degree of a differential form and so we cannot use the above proof to reach the desired conclusion. However, with a more refined index theorem [28, Theorem 6.24], one can indeed conclude that $0\in\sigma\left(\triangle_{\frac{\dim(M)}{2}}\right)$ if $\dim(M)$ is even and that $0\in\sigma\left(\triangle_{\frac{\dim(M)\pm1}{2}}\right)$ if $\dim(M)$ is odd.

2. If M is an irreducible noncompact globally symmetric space G/K, with $G = \mathrm{Isom}(M)$ and K a maximal compact subgroup, then one can say more about the bottom of the spectrum. If $\mathrm{rk}(G) = \mathrm{rk}(K)$ then $\mathrm{Ker}\left(\triangle_{\frac{dim(M)}{2}}\right)$ is infinite-dimensional and the spectrum of $\triangle$ is bounded away from zero otherwise. If $\mathrm{rk}(G) > \mathrm{rk}(K)$ then $\mathrm{Ker}(\triangle) = 0$ and $0 \in \sigma(\triangle_p)$ if and only if

$$p \in \left[\frac{dim(M)}{2} - \frac{rk(G) - rk(K)}{2}, \frac{dim(M)}{2} + \frac{rk(G) - rk(K)}{2}\right]$$

[19, Section VIIB].

Finally, we state a result about uniformly contractible Riemannian manifolds.

DEFINITION 6 [15, p. 29]. *A metric space Z has finite asymptotic dimension if there is an integer n such that for any $r > 0$, there is a covering $Z = \bigcup_{i \in I} C_i$ of Z by subsets of uniformly bounded diameter so that no metric ball of radius r in Z intersects more than $n+1$ elements of $\{C_i\}_{i \in I}$. The smallest such integer n is called the asymptotic dimension* $\mathrm{asdim}_+(Z)$ *of Z.*

PROPOSITION 8 (Yu [33]). *If M is a uniformly contractible Riemannian manifold with finite asymptotic dimension then $0 \in \sigma(\triangle_p)$ for some p.*

The proof of Proposition 8 uses methods of coarse index theory [28].

4. Very Low Dimensions

In this section we show that the answer to the zero-in-the-spectrum question is "yes" for one-dimensional simplicial complexes and two-dimensional Riemannian manifolds.

4.1 One Dimension

As a one-dimensional manifold is either S^1 or $\mathbf{R}$, zero is clearly in the spectrum.

A more interesting problem is to consider a connected one-dimensional simplicial complex K. Let V be the set of vertices of K and let E be the set of oriented edges of K. That is, an element e of E consists of an edge of K and an ordering (s_e, t_e) of ∂e. We let $-e$ denote the same edge with the