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2. If M is an irreducible noncompact globally symmetric space G/K, with $G = \mathrm{Isom}(M)$ and K a maximal compact subgroup, then one can say more about the bottom of the spectrum. If $\mathrm{rk}(G) = \mathrm{rk}(K)$ then $\mathrm{Ker}\left(\triangle_{\frac{dim(M)}{2}}\right)$ is infinite-dimensional and the spectrum of $\triangle$ is bounded away from zero otherwise. If $\mathrm{rk}(G) > \mathrm{rk}(K)$ then $\mathrm{Ker}(\triangle) = 0$ and $0 \in \sigma(\triangle_p)$ if and only if

$$p \in \left[\frac{dim(M)}{2} - \frac{rk(G) - rk(K)}{2}, \frac{dim(M)}{2} + \frac{rk(G) - rk(K)}{2}\right]$$

[19, Section VIIB].

Finally, we state a result about uniformly contractible Riemannian manifolds.

DEFINITION 6 [15, p. 29]. *A metric space Z has finite asymptotic dimension if there is an integer n such that for any $r > 0$, there is a covering $Z = \bigcup_{i \in I} C_i$ of Z by subsets of uniformly bounded diameter so that no metric ball of radius r in Z intersects more than $n + 1$ elements of $\{C_i\}_{i \in I}$. The smallest such integer n is called the asymptotic dimension* $\mathrm{asdim}_+(Z)$ *of Z.*

PROPOSITION 8 (Yu [33]). *If M is a uniformly contractible Riemannian manifold with finite asymptotic dimension then $0 \in \sigma(\triangle_p)$ for some p.*

The proof of Proposition 8 uses methods of coarse index theory [28].

4. VERY LOW DIMENSIONS

In this section we show that the answer to the zero-in-the-spectrum question is "yes" for one-dimensional simplicial complexes and two-dimensional Riemannian manifolds.

4.1 ONE DIMENSION

As a one-dimensional manifold is either S^1 or $\mathbf{R}$, zero is clearly in the spectrum.

A more interesting problem is to consider a connected one-dimensional simplicial complex K. Let V be the set of vertices of K and let E be the set of oriented edges of K. That is, an element e of E consists of an edge of K and an ordering (s_e, t_e) of ∂e. We let $-e$ denote the same edge with the reverse ordering of ∂e. For $x \in V$, let m_x denote the number of unoriented edges of which x is a boundary. We assume that $m_x < \infty$ for all x. Put

$$C^0(K) = \{f : V \to \mathbf{C} \text{ such that } \sum_{x \in V} m_x |f(x)|^2 < \infty\},$$

$$\text{(4.1)} \qquad C^1(K) = \{F : E \to \mathbf{C} \text{ such that } F(-e) = -F(e) \text{ and } \frac{1}{2} \sum_{e \in E} |F(e)|^2 < \infty\}.$$

Then $C^0(K)$ and $C^1(K)$ are Hilbert spaces. The weighting used to define $C^0(K)$ is natural in certain respects [8].

There is a bounded operator $d : C^0(K) \to C^1(K)$ given by $(df)(e) = f(t_e) - f(s_e)$. Define the Laplace-Beltrami operators by $\triangle_0 = d^*d$ and $\triangle_1 = dd^*$. An element of $\text{Ker}(\triangle_1)$ is an $F \in C^1(K)$ such that for each vertex x the total current flowing into x vanishes, i.e. $\sum_{e \in E : t_e = x} F(e) = 0$.

The next proposition is essentially due to Gromov [15, p. 236], who proved it in the case when $\{m_x\}_{x \in V}$ is bounded.

PROPOSITION 9. *$0 \in \sigma(\triangle_0)$ or $0 \in \sigma(\triangle_1)$.*

Proof. As the nonzero spectra of d^*d and dd^* are the same, for our purposes it suffices to consider $\sigma(\triangle_0)$ and $\text{Ker}(\triangle_1)$. We argue by contradiction. Suppose that $0 \notin \sigma(\triangle_0)$ and $\text{Ker}(\triangle_1) = 0$. First, K must be infinite, as otherwise $\text{Ker}(\triangle_0) \neq 0$. Second, K must be a tree, as if K had a loop then we could create a nonzero element of $\text{Ker}(\triangle_1)$ by letting a current of unit strength flow around the loop.

We now show that K has lots of branching. For $x, y \in V$, let $[x, y]$ be the geodesic arc from x to y and let (x, y) be its interior. Let $d(x, y)$ be the number of edges in $[x, y]$.

LEMMA 5. *There is a constant $L > 0$ such that if $d(x, y) > L$ then there is an infinite subtree of K which intersects (x, y) but does not contain x or y.*

Proof. If the lemma is not true then for all $N > 1$, there are vertices x and y such that $d(x, y) > N$ but there are no infinite subtrees as in the statement of the lemma. In other words, the connected component C of $K - \{x\} - \{y\}$ which contains (x, y) is finite. As K is a tree, x is only connected to the vertices in C by a single edge, and similarly for y (see Fig. 5). Define $f \in C^0(K)$ by

$$f(v) = \begin{cases} 1 & \text{if } v \in C, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Then

$$\frac{\langle df, df\rangle}{\langle f, f\rangle} \leq \frac{2}{2\big(d(x,y)-1\big)} \leq \frac{1}{N}. \tag{4.3}$$

As N can be taken arbitrarily large, this contradicts the assumption that $0 \notin \sigma(\triangle_0)$. □

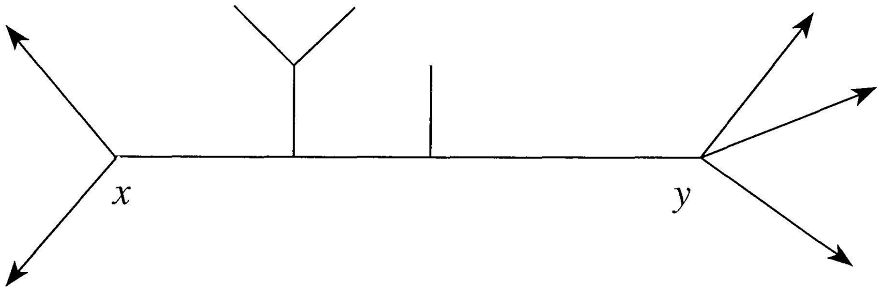

FIGURE 5

It follows that K contains a subtree K' which is topologically equivalent to an infinite triadic tree, with the distances between branchings at most L (see Fig. 6). We can create a nonzero square-integrable harmonic 1-cochain F' on K' by letting a unit current flow through it, as in Fig. 6. Let $F \in C^1(K)$ be the extension of F' by zero to K. If x is a vertex of K' then the total current flowing into x is still zero, as no new current comes in along the edges of $K - K'$. Thus $\mathrm{Ker}(\triangle_1) \neq 0$, which is a contradiction. □

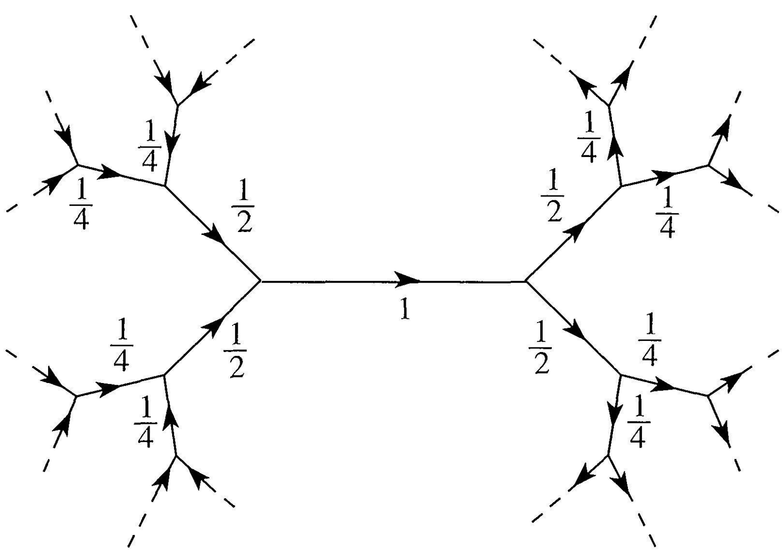

FIGURE 6

4.2 Two Dimensions

PROPOSITION 10 (Lott, Dodziuk). *The answer to the zero-in-the-spectrum question is "yes" if M is a two-dimensional manifold.*

Proof. The Hodge decomposition gives

$$\Lambda^0(M) = \operatorname{Ker}(\triangle_0) \oplus \Lambda^0(M)/\operatorname{Ker}(d), \tag{4.4}$$

$$\Lambda^1(M) = \operatorname{Ker}(\triangle_1) \oplus \overline{d\Lambda^0(M)} \oplus \overline{*d\Lambda^0(M)}, \tag{4.5}$$

$$\Lambda^2(M) = *\operatorname{Ker}(\triangle_0) \oplus *\big(\Lambda^0(M)/\operatorname{Ker}(d)\big). \tag{4.6}$$

Thus it is enough to look at

$$\operatorname{Ker}(\triangle_0), \quad \operatorname{Ker}(\triangle_1) \quad \text{and} \quad \sigma\left(\triangle_0 \text{ on } \Lambda^0(M)/\operatorname{Ker}(d)\right).$$

We argue by contradiction. Assume that zero is not in the spectrum. By Proposition 4, $\operatorname{Im}\big(\mathrm{H}^1_c(M) \to \mathrm{H}^1(M)\big) = 0$. Thus M must be planar, in the sense of either of the following two equivalent conditions:

1. Any simple closed curve in M separates it into two pieces.
2. M is diffeomorphic to the complement of a closed subset of S^2.

As $\operatorname{Ker}(\triangle_0) = 0$, M cannot be S^2. By Proposition 5, the possible existence of nonzero square-integrable harmonic 1-forms on M only depends on the underlying Riemann surface coming from the Riemannian metric on M.

We recall some notions from Riemann surface theory [1]. A function $f \in C^\infty(M)$ is *superharmonic* if $\triangle_0 f > 0$. (This is a conformally-invariant statement.) The Riemann surface underlying M is *hyperbolic* if it has a positive superharmonic function and *parabolic* otherwise. If M is planar and hyperbolic then there is a nonconstant harmonic function $f \in C^\infty(M)$ such that $\int_M df \wedge *df < \infty$ [1, p. 208]. Then df would be a nonzero element of $\operatorname{Ker}(\triangle_1)$. Thus M must be parabolic.

Put $\lambda_0 = \inf\big(\sigma(\triangle_0)\big)$. Choose some λ such that $0 < \lambda < \lambda_0$. Then there is a positive $f \in C^\infty(M)$ (not square-integrable !) such that $\triangle_0 f = \lambda f$ [31, Theorem 2.1]. However, this contradicts the parabolicity of M. □

We do not know of any result analogous to Proposition 10 for general two-dimensional simplicial complexes, say uniformly finite. See, however, Subsection 5.2.