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2. If  $M$  is an irreducible noncompact globally symmetric space  $G/K$ , with  $G = \text{Isom}(M)$  and  $K$  a maximal compact subgroup, then one can say more about the bottom of the spectrum. If  $\text{rk}(G) = \text{rk}(K)$  then  $\text{Ker} \left( \Delta_{\frac{\dim(M)}{2}} \right)$  is infinite-dimensional and the spectrum of  $\Delta$  is bounded away from zero otherwise. If  $\text{rk}(G) > \text{rk}(K)$  then  $\text{Ker}(\Delta) = 0$  and  $0 \in \sigma(\Delta_p)$  if and only if

$$p \in \left[ \frac{\dim(M)}{2} - \frac{\text{rk}(G) - \text{rk}(K)}{2}, \frac{\dim(M)}{2} + \frac{\text{rk}(G) - \text{rk}(K)}{2} \right]$$

[19, Section VIIB].

Finally, we state a result about uniformly contractible Riemannian manifolds.

DEFINITION 6 [15, p. 29]. *A metric space  $Z$  has finite asymptotic dimension if there is an integer  $n$  such that for any  $r > 0$ , there is a covering  $Z = \bigcup_{i \in I} C_i$  of  $Z$  by subsets of uniformly bounded diameter so that no metric ball of radius  $r$  in  $Z$  intersects more than  $n + 1$  elements of  $\{C_i\}_{i \in I}$ . The smallest such integer  $n$  is called the asymptotic dimension  $\text{asdim}_+(Z)$  of  $Z$ .*

PROPOSITION 8 (Yu [33]). *If  $M$  is a uniformly contractible Riemannian manifold with finite asymptotic dimension then  $0 \in \sigma(\Delta_p)$  for some  $p$ .*

The proof of Proposition 8 uses methods of coarse index theory [28].

#### 4. VERY LOW DIMENSIONS

In this section we show that the answer to the zero-in-the-spectrum question is “yes” for one-dimensional simplicial complexes and two-dimensional Riemannian manifolds.

##### 4.1 ONE DIMENSION

As a one-dimensional manifold is either  $S^1$  or  $\mathbf{R}$ , zero is clearly in the spectrum.

A more interesting problem is to consider a connected one-dimensional simplicial complex  $K$ . Let  $V$  be the set of vertices of  $K$  and let  $E$  be the set of oriented edges of  $K$ . That is, an element  $e$  of  $E$  consists of an edge of  $K$  and an ordering  $(s_e, t_e)$  of  $\partial e$ . We let  $-e$  denote the same edge with the

reverse ordering of  $\partial e$ . For  $x \in V$ , let  $m_x$  denote the number of unoriented edges of which  $x$  is a boundary. We assume that  $m_x < \infty$  for all  $x$ . Put

$$C^0(K) = \{f : V \rightarrow \mathbf{C} \text{ such that } \sum_{x \in V} m_x |f(x)|^2 < \infty\},$$

$$(4.1) \quad C^1(K) = \{F : E \rightarrow \mathbf{C} \text{ such that } F(-e) = -F(e) \text{ and } \frac{1}{2} \sum_{e \in E} |F(e)|^2 < \infty\}.$$

Then  $C^0(K)$  and  $C^1(K)$  are Hilbert spaces. The weighting used to define  $C^0(K)$  is natural in certain respects [8].

There is a bounded operator  $d : C^0(K) \rightarrow C^1(K)$  given by  $(df)(e) = f(t_e) - f(s_e)$ . Define the Laplace-Beltrami operators by  $\Delta_0 = d^*d$  and  $\Delta_1 = dd^*$ . An element of  $\text{Ker}(\Delta_1)$  is an  $F \in C^1(K)$  such that for each vertex  $x$  the total current flowing into  $x$  vanishes, i.e.  $\sum_{e \in E: t_e=x} F(e) = 0$ .

The next proposition is essentially due to Gromov [15, p. 236], who proved it in the case when  $\{m_x\}_{x \in V}$  is bounded.

PROPOSITION 9.  $0 \in \sigma(\Delta_0)$  or  $0 \in \sigma(\Delta_1)$ .

*Proof.* As the nonzero spectra of  $d^*d$  and  $dd^*$  are the same, for our purposes it suffices to consider  $\sigma(\Delta_0)$  and  $\text{Ker}(\Delta_1)$ . We argue by contradiction. Suppose that  $0 \notin \sigma(\Delta_0)$  and  $\text{Ker}(\Delta_1) = 0$ . First,  $K$  must be infinite, as otherwise  $\text{Ker}(\Delta_0) \neq 0$ . Second,  $K$  must be a tree, as if  $K$  had a loop then we could create a nonzero element of  $\text{Ker}(\Delta_1)$  by letting a current of unit strength flow around the loop.

We now show that  $K$  has lots of branching. For  $x, y \in V$ , let  $[x, y]$  be the geodesic arc from  $x$  to  $y$  and let  $(x, y)$  be its interior. Let  $d(x, y)$  be the number of edges in  $[x, y]$ .

LEMMA 5. *There is a constant  $L > 0$  such that if  $d(x, y) > L$  then there is an infinite subtree of  $K$  which intersects  $(x, y)$  but does not contain  $x$  or  $y$ .*

*Proof.* If the lemma is not true then for all  $N > 1$ , there are vertices  $x$  and  $y$  such that  $d(x, y) > N$  but there are no infinite subtrees as in the statement of the lemma. In other words, the connected component  $C$  of  $K - \{x\} - \{y\}$  which contains  $(x, y)$  is finite. As  $K$  is a tree,  $x$  is only connected to the vertices in  $C$  by a single edge, and similarly for  $y$  (see Fig. 5). Define  $f \in C^0(K)$  by

$$(4.2) \quad f(v) = \begin{cases} 1 & \text{if } v \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(4.3) \quad \frac{\langle df, df \rangle}{\langle f, f \rangle} \leq \frac{2}{2(d(x, y) - 1)} \leq \frac{1}{N}.$$

As  $N$  can be taken arbitrarily large, this contradicts the assumption that  $0 \notin \sigma(\Delta_0)$ .  $\square$



FIGURE 5

It follows that  $K$  contains a subtree  $K'$  which is topologically equivalent to an infinite triadic tree, with the distances between branchings at most  $L$  (see Fig. 6). We can create a nonzero square-integrable harmonic 1-cochain  $F'$  on  $K'$  by letting a unit current flow through it, as in Fig. 6. Let  $F \in C^1(K)$  be the extension of  $F'$  by zero to  $K$ . If  $x$  is a vertex of  $K'$  then the total current flowing into  $x$  is still zero, as no new current comes in along the edges of  $K - K'$ . Thus  $\text{Ker}(\Delta_1) \neq 0$ , which is a contradiction.  $\square$

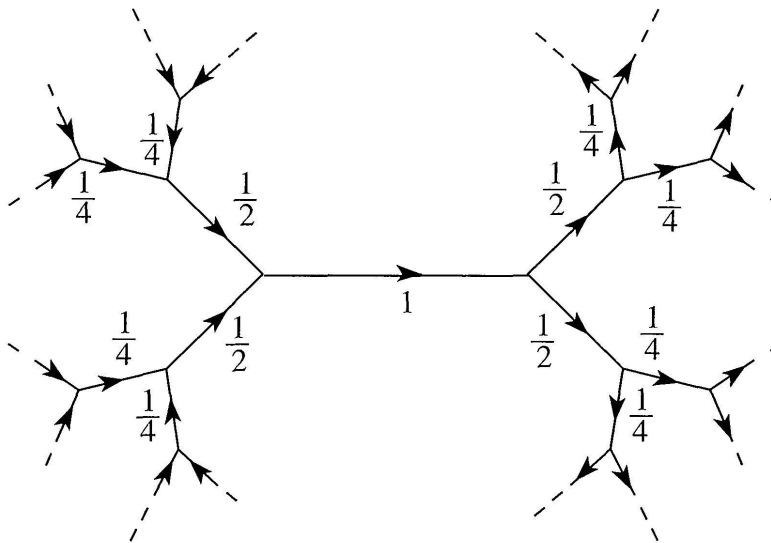


FIGURE 6