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In what follows, Γ will denote a finitely-presented group. Given a presentation of Γ, there is an associated 2-dimensional CW-complex K which we call the *presentation complex*. To form it, make a bouquet of circles indexed by the generators of Γ. Attach 2-cells based on the relations of Γ. (We allow trivial or repeated relations in the presentation.) This is the presentation complex.

DEFINITION 7. *Put* $b_0^{(2)}(\Gamma) = b_0^{(2)}(K)$, $b_1^{(2)}(\Gamma) = b_1^{(2)}(K)$, $\alpha_1(\Gamma) = \alpha_1(K)$ *and* $\alpha_2(\Gamma) = \alpha_2(K)$.

By Property 4 above, Definition 7 makes sense in that the choice of presentation of Γ does not matter.

As the invariants $b_0^{(2)}(\Gamma)$, $b_1^{(2)}(\Gamma)$, $\alpha_1(\Gamma)$ and $\alpha_2(\Gamma)$ will play an important role, let us state explicitly what they measure. First, from Property 5, $b_0^{(2)}(\Gamma)$ tells us whether or not Γ is infinite. In general, $b_0^{(2)}(\Gamma) = \frac{1}{|\Gamma|}$. Next, from Property 1, $b_1^{(2)}(\Gamma)$ tells us whether or not M has square-integrable harmonic 1-forms (or $\widetilde{K}$ has square-integrable harmonic 1-cochains). From Property 2, $\alpha_1(\Gamma)$ tells us whether or not the Laplacian $\triangle_0$, acting on functions on M, has a gap in its spectrum away from zero. In fact, Property 6 is just a restatement of Corollary 3. Finally, from Property 2, $\alpha_2(\Gamma)$ tells us whether or not the spectrum of the Laplacian on $\Lambda^1(M)/\operatorname{Ker}(d)$ goes down to zero.

5.1 Big and Small Groups

Let us first introduce a convenient terminology for the purposes of the present paper.

DEFINITION 8. *The group* Γ *is big if it is nonamenable,* $b_1^{(2)}(\Gamma) = 0$ *and* $\alpha_2(\Gamma) = \infty^+$. *Otherwise,* Γ *is small.*

We recall that $\triangle_p$ denotes the Laplace-Beltrami operator on the universal cover M.

PROPOSITION 11. *Let X and M be as above. The group* $\pi_1(X)$ *is small if and only if* $0 \in \sigma(\triangle_0)$ *or* $0 \in \sigma(\triangle_1)$.

Proof. This follows immediately from Properties 1, 2, 4, 5 and 6 above. $\square$

The question arises as to which groups are big and which are small. Clearly any amenable group is small.

PROPOSITION 12. *Fundamental groups of compact surfaces are small.*

Proof. Suppose that Σ is a compact surface and $\Gamma = \pi_1(\Sigma)$. If Σ has boundary then Γ is a free group F_j on some number j of generators. If $j = 0$ or $j = 1$ then Γ is amenable. If $j > 1$ then $b_1^{(2)}(\Gamma) = j - 1 > 0$.

Suppose now that Σ is closed. If $\chi(\Sigma) \geq 0$ then Γ is amenable. If $\chi(\Sigma) < 0$ then $b_1^{(2)}(\Gamma) = -\chi(\Sigma) > 0$. $\square$

We now extend Proposition 12 to 3-manifold groups. We use some facts about compact connected 3-manifolds Y, possibly with boundary. (See, for example, [21, Section 6]). Again, all of our manifolds are assumed to be oriented. First, Y has a decomposition as a connected sum $Y = Y_1 \# Y_2 \# \ldots \# Y_r$ of *prime* 3-manifolds. A prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy-equivalent to a Seifert, Haken or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known and it is likely that there are none.

PROPOSITION 13 (Lott-Lück). *Suppose that Y is a compact connected oriented 3-manifold, possibly with boundary, none of whose prime factors are exceptional. Then $\pi_1(Y)$ is small.*

Proof. We argue by contradiction. Suppose that $\pi_1(Y)$ is big. First, $\pi_1(Y)$ must be infinite. If ∂Y has any connected components which are 2-spheres then we can cap them off with 3-balls without changing $\pi_1(Y)$. So we can assume that ∂Y does not have any 2-sphere components. In particular, $\chi(Y) = \frac{1}{2}\chi(\partial Y) \leq 0$. From [21, Theorem 0.1.1],

$$b_1^{(2)}(Y) = (r-1) - \sum_{i=1}^{r} \frac{1}{|\pi_1(Y_i)|} - \chi(Y). \tag{5.3}$$

As this must vanish, we have $\chi(Y) = 0$ and either

1. $\{|\pi_1(Y_i)|\}_{i=1}^r = \{2, 2, 1, \ldots, 1\}$ or
2. $\{|\pi_1(Y_i)|\}_{i=1}^r = \{\infty, 1, \ldots, 1\}$.

It follows that ∂Y is empty or a disjoint union of 2-tori. As there are no 2-spheres in ∂Y, if $|\pi_1(Y_i)| = 1$ then Y_i is a homotopy 3-sphere. Thus Y is homotopy-equivalent to either

1. $\mathbf{R}P^3 \# \mathbf{R}P^3$ or
2. A prime 3-manifold Y' with infinite fundamental group whose boundary is empty or a disjoint union of 2-tori.

If Y is homotopy-equivalent to $\mathbf{R}P^3\#\mathbf{R}P^3$ then $\pi_1(Y)$ is amenable, which is a contradiction. So we must be in the second case. Using Property 3, we may assume that $Y = Y'$. Then as Y is prime, it follows from [24, Chapter 1] that either $Y = S^1 \times D^2$ or Y has incompressible (or empty) boundary. If $Y = S^1 \times D^2$ then $\pi_1(Y)$ is amenable. If Y has incompressible (or empty) boundary then from [21, Theorem 0.1.5], $\alpha_2(Y) \leq 2$ unless Y is a closed 3-manifold with an $\mathbf{R}^3$, $\mathbf{R} \times S^2$ or *Sol* geometric structure. In the latter cases, Γ is amenable. Thus in any case, we get a contradiction. □

The next proposition gives examples of big groups.

PROPOSITION 14.

1. *A product of two nonamenable groups is big.*
2. *If Y is a closed nonpositively-curved locally symmetric space of dimension greater than three, with no Euclidean factors in $\widetilde{Y}$, then $\pi_1(Y)$ is big.*

Proof. 1. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1 and Γ_2 nonamenable. Then Γ is nonamenable. Let K_1 and K_2 be presentation complexes with fundamental groups Γ_1 and Γ_2, respectively. Put $K = K_1 \times K_2$. Then $\Gamma = \pi_1(K)$. Let $\triangle_p(\widetilde{K})$, $\triangle_p(\widetilde{K_1})$ and $\triangle_p(\widetilde{K_2})$ denote the Laplace-Beltrami operator on p-cochains on $\widetilde{K}$, $\widetilde{K_1}$ and $\widetilde{K_2}$, respectively, as defined in Subsection 5.2 below. Then

$$\begin{aligned}\inf\big(\sigma\big(\triangle_1(\widetilde{K})\big)\big) = \min\big(&\inf\big(\sigma\big(\triangle_1(\widetilde{K_1})\big)\big) + \inf\big(\sigma\big(\triangle_0(\widetilde{K_2})\big)\big),\\ &\inf\big(\sigma\big(\triangle_0(\widetilde{K_1})\big)\big) + \inf\big(\sigma\big(\triangle_1(\widetilde{K_2})\big)\big)\big) > 0.\end{aligned} \tag{5.4}$$

Using Proposition 11, the first part of the proposition follows.

2. If $\widetilde{Y}$ is irreducible then part 2. of the proposition follows from the second remark after Proposition 7. If $\widetilde{Y}$ is reducible then we can use an argument similar to (5.4). □

REMARK. Let Γ be an infinite finitely-presented discrete group with Kazhdan's property T. From [6, p. 47], $\mathrm{H}^1\big(\Gamma; l^2(\Gamma)\big) = 0$. This implies that Γ is nonamenable and $b_1^{(2)}(\Gamma) = 0$. We do not know if it is necessarily true that $\alpha_2(\Gamma) = \infty^+$.

5.2 Two and Three Dimensions

In this subsection we relate the zero-in-the-spectrum question to a question in combinatorial group theory. Let K be a finite connected 2-dimensional