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If Y is homotopy-equivalent to $\mathbb{R}P^3 \#\mathbb{R}P^3$ then $\pi_1(Y)$ is amenable, which is a contradiction. So we must be in the second case. Using Property 3, we may assume that Y = Y'. Then as Y is prime, it follows from [24, Chapter 1] that either $Y = S^1 \times D^2$ or Y has incompressible (or empty) boundary. If $Y = S^1 \times D^2$ then $\pi_1(Y)$ is amenable. If Y has incompressible (or empty) boundary then from [21, Theorem 0.1.5], $\alpha_2(Y) \leq 2$ unless Y is a closed 3-manifold with an \mathbb{R}^3 , $\mathbb{R} \times S^2$ or Sol geometric structure. In the latter cases, Γ is amenable. Thus in any case, we get a contradiction. \Box

The next proposition gives examples of big groups.

PROPOSITION 14.

- 1. A product of two nonamenable groups is big.
- 2. If Y is a closed nonpositively-curved locally symmetric space of dimension greater than three, with no Euclidean factors in \tilde{Y} , then $\pi_1(Y)$ is big.

Proof. 1. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1 and Γ_2 nonamenable. Then Γ is nonamenable. Let K_1 and K_2 be presentation complexes with fundamental groups Γ_1 and Γ_2 , respectively. Put $K = K_1 \times K_2$. Then $\Gamma = \pi_1(K)$. Let $\Delta_p(\widetilde{K})$, $\Delta_p(\widetilde{K_1})$ and $\Delta_p(\widetilde{K_2})$ denote the Laplace-Beltrami operator on p-cochains on \widetilde{K} , $\widetilde{K_1}$ and $\widetilde{K_2}$, respectively, as defined in Subsection 5.2 below. Then

(5.4)
$$\inf\left(\sigma\left(\bigtriangleup_{1}(\widetilde{K})\right)\right) = \min\left(\inf\left(\sigma\left(\bigtriangleup_{1}(\widetilde{K_{1}})\right)\right) + \inf\left(\sigma\left(\bigtriangleup_{0}(\widetilde{K_{2}})\right)\right), \\ \inf\left(\sigma\left(\bigtriangleup_{0}(\widetilde{K_{1}})\right)\right) + \inf\left(\sigma\left(\bigtriangleup_{1}(\widetilde{K_{2}})\right)\right)\right) > 0.$$

Using Proposition 11, the first part of the proposition follows.

2. If \widetilde{Y} is irreducible then part 2. of the proposition follows from the second remark after Proposition 7. If \widetilde{Y} is reducible then we can use an argument similar to (5.4).

REMARK. Let Γ be an infinite finitely-presented discrete group with Kazhdan's property T. From [6, p. 47], $H^1(\Gamma; l^2(\Gamma)) = 0$. This implies that Γ is nonamenable and $b_1^{(2)}(\Gamma) = 0$. We do not know if it is necessarily true that $\alpha_2(\Gamma) = \infty^+$.

5.2 Two and Three Dimensions

In this subsection we relate the zero-in-the-spectrum question to a question in combinatorial group theory. Let K be a finite connected 2-dimensional *CW*-complex. Let \widetilde{K} be its universal cover. Let $C^*(\widetilde{K})$ denote the Hilbert space of square-integrable cellular cochains on \widetilde{K} . There is a cochain complex

(5.5)
$$0 \longrightarrow C^{0}(\widetilde{K}) \xrightarrow{d_{0}} C^{1}(\widetilde{K}) \xrightarrow{d_{1}} C^{2}(\widetilde{K}) \longrightarrow 0.$$

Define the Laplace-Beltrami operators by $\triangle_0 = d_0^* d_0$, $\triangle_1 = d_0 d_0^* + d_1^* d_1$ and $\triangle_2 = d_1 d_1^*$. These are bounded self-adjoint operators and so we can talk about zero being in the spectrum of \widetilde{K} .

PROPOSITION 15. Zero is not in the spectrum of \tilde{K} if and only if $\pi_1(K)$ is big and $\chi(K) = 0$.

Proof. Suppose that zero is not in the spectrum of \widetilde{K} . From the analog of Proposition 11, Γ must be big. Furthermore, from Properties 1 and 7, $\chi(K) = 0$.

Now suppose that $\pi_1(K)$ is big and $\chi(K) = 0$. From the analog of Proposition 11, $0 \notin \sigma(\triangle_0)$ and $0 \notin \sigma(\triangle_1)$. In particular, $\operatorname{Ker}(\triangle_0) = \operatorname{Ker}(\triangle_1) = 0$. From Properties 1 and 7, $\operatorname{Ker}(\triangle_2) = 0$. As $C^2(\widetilde{K}) = \operatorname{Ker}(\triangle_2) \oplus \overline{d_1C^1(\widetilde{K})}$, we conclude that $0 \notin \sigma(\triangle_2)$. \Box

Let Γ be a finitely-presented group. Consider a fixed presentation of Γ consisting of g generators and r relations. Let K be the corresponding presentation complex. Then $\chi(K) = 1 - g + r$. Thus zero is not in the spectrum of \widetilde{K} if and only if $\pi_1(K)$ is big and g - r = 1.

Recall that the *deficiency* def(Γ) is defined to be the maximum, over all finite presentations of Γ , of g - r. If $b_1^{(2)}(\Gamma) = 0$ then from the equation

(5.6)
$$\chi(K) = 1 - g + r = b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(K),$$

we obtain def(Γ) ≤ 1 . This is the case, for example, when Γ is big or when Γ is amenable [5].

As any finite connected 2-dimensional *CW*-complex is homotopyequivalent to a presentation complex, it follows from Proposition 15 that the answer to the zero-in-the-spectrum question is "yes" for universal covers of such complexes if and only if the following conjecture is true.

CONJECTURE 1. If Γ is a big group then def $(\Gamma) \leq 0$.

REMARK. If $\pi_1(K)$ has property T then the ordinary first Betti number of K vanishes [6], and so $\chi(K) = 1 + b_2(K) > 0$. Thus zero lies in the spectrum of \widetilde{K} .

Now let Y be a 3-manifold satisfying the conditions of Proposition 13. If $\partial Y \neq \emptyset$, we define \triangle_p on \widetilde{Y} using absolute boundary conditions on $\partial \widetilde{Y}$.

PROPOSITION 16. Zero lies in the spectrum of \tilde{Y} .

Proof. This is a consequence of Propositions 11 and 13. \Box

5.3 FOUR DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If M is a Riemannian 4-manifold then the Hodge decomposition gives

(5.7)
$$\Lambda^{0}(M) = \operatorname{Ker}(\Delta_{0}) \oplus \Lambda^{0}(M) / \operatorname{Ker}(d),$$
$$\Lambda^{1}(M) = \operatorname{Ker}(\Delta_{1}) \oplus \overline{d\Lambda^{0}(M)} \oplus \Lambda^{1}(M) / \operatorname{Ker}(d),$$
$$\Lambda^{2}(M) = \operatorname{Ker}(\Delta_{2}) \oplus \overline{d\Lambda^{1}(M)} \oplus *\overline{d\Lambda^{1}(M)},$$
$$\Lambda^{3}(M) = *\operatorname{Ker}(\Delta_{1}) \oplus *\overline{d\Lambda^{0}(M)} \oplus *(\Lambda^{1}(M) / \operatorname{Ker}(d)),$$
$$\Lambda^{4}(M) = *\operatorname{Ker}(\Delta_{0}) \oplus *(\Lambda^{0}(M) / \operatorname{Ker}(d)).$$

Thus for the zero-in-the-spectrum question, it is enough to consider Ker(\triangle_0), Ker(\triangle_1), $\sigma(\triangle_0 \text{ on } \Lambda^0 / \text{Ker}(d))$, $\sigma(\triangle_1 \text{ on } \Lambda^1 / \text{Ker}(d))$ and Ker(\triangle_2).

Let Γ be a finitely-presented group. Recall that Γ is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of Γ . Embed the resulting presentation complex in \mathbf{R}^5 and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds X with fundamental group Γ . Given X, we have $\chi(X\#\mathbb{C}P^2) = \chi(X) + 1$. Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

(5.8)
$$\{\chi(X): X \text{ is a closed connected oriented 4-manifold with} \\ \pi_1(X) = \Gamma\} = \{n \in \mathbb{Z} : n \ge q(\Gamma)\}$$

for some $q(\Gamma)$. A priori $q(\Gamma) \in \mathbb{Z} \cup \{-\infty\}$, but in fact $q(\Gamma) \in \mathbb{Z}$ [17, Theorem 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of $q(\Gamma)$.

Suppose that $\pi_1(X) = \Gamma$. From Properties 4, 7 and 8 above,

(5.9)
$$\chi(X) = 2b_0^{(2)}(\Gamma) - 2b_1^{(2)}(\Gamma) + b_2^{(2)}(X).$$

In particular, if $b_1^{(2)}(\Gamma) = 0$ then $\chi(X) \ge 0$ and so $q(\Gamma) \ge 0$. This is the case, for example, when Γ is big or when Γ is amenable [5].