Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 42 (1996)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE ZERO-IN-THE-SPECTRUM QUESTION

Autor: LOTT, John

Kapitel: 5.2 Two and Three Dimensions

DOI: https://doi.org/10.5169/seals-87882

If Y is homotopy-equivalent to $\mathbf{R}P^3\#\mathbf{R}P^3$ then $\pi_1(Y)$ is amenable, which is a contradiction. So we must be in the second case. Using Property 3, we may assume that $Y = Y'$. Then as Y is prime, it follows from [24, Chapter 1] that either $Y = S^1 \times D^2$ or Y has incompressible (or empty) boundary. If $Y = S^1 \times D^2$ then $\pi_1(Y)$ is amenable. If Y has incompressible (or empty) boundary then from [21, Theorem 0.1.5], $\alpha_2(Y) \leq 2$ unless Y is a closed 3-manifold with an $\mathbf{R}^3$, $\mathbf{R} \times S^2$ or *Sol* geometric structure. In the latter cases, Γ is amenable. Thus in any case, we get a contradiction. □

The next proposition gives examples of big groups.

PROPOSITION 14.

1. *A product of two nonamenable groups is big.*
2. *If Y is a closed nonpositively-curved locally symmetric space of dimension greater than three, with no Euclidean factors in $\widetilde{Y}$, then $\pi_1(Y)$ is big.*

Proof. 1. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1 and Γ_2 nonamenable. Then Γ is nonamenable. Let K_1 and K_2 be presentation complexes with fundamental groups Γ_1 and Γ_2, respectively. Put $K = K_1 \times K_2$. Then $\Gamma = \pi_1(K)$. Let $\triangle_p(\widetilde{K})$, $\triangle_p(\widetilde{K_1})$ and $\triangle_p(\widetilde{K_2})$ denote the Laplace-Beltrami operator on p-cochains on $\widetilde{K}$, $\widetilde{K_1}$ and $\widetilde{K_2}$, respectively, as defined in Subsection 5.2 below. Then

$$\begin{aligned}\inf\big(\sigma\big(\triangle_1(\widetilde{K})\big)\big) = \min\big(&\inf\big(\sigma\big(\triangle_1(\widetilde{K_1})\big)\big) + \inf\big(\sigma\big(\triangle_0(\widetilde{K_2})\big)\big),\\ &\inf\big(\sigma\big(\triangle_0(\widetilde{K_1})\big)\big) + \inf\big(\sigma\big(\triangle_1(\widetilde{K_2})\big)\big)\big) > 0.\end{aligned} \tag{5.4}$$

Using Proposition 11, the first part of the proposition follows.

2. If $\widetilde{Y}$ is irreducible then part 2. of the proposition follows from the second remark after Proposition 7. If $\widetilde{Y}$ is reducible then we can use an argument similar to (5.4). □

REMARK. Let Γ be an infinite finitely-presented discrete group with Kazhdan's property T. From [6, p. 47], $\mathrm{H}^1\big(\Gamma; l^2(\Gamma)\big) = 0$. This implies that Γ is nonamenable and $b_1^{(2)}(\Gamma) = 0$. We do not know if it is necessarily true that $\alpha_2(\Gamma) = \infty^+$.

5.2 TWO AND THREE DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question in combinatorial group theory. Let K be a finite connected 2-dimensional

CW-complex. Let $\widetilde{K}$ be its universal cover. Let $C^*(\widetilde{K})$ denote the Hilbert space of square-integrable cellular cochains on $\widetilde{K}$. There is a cochain complex

$$0 \longrightarrow C^0(\widetilde{K}) \xrightarrow{d_0} C^1(\widetilde{K}) \xrightarrow{d_1} C^2(\widetilde{K}) \longrightarrow 0. \tag{5.5}$$

Define the Laplace-Beltrami operators by $\triangle_0 = d_0^* d_0$, $\triangle_1 = d_0 d_0^* + d_1^* d_1$ and $\triangle_2 = d_1 d_1^*$. These are bounded self-adjoint operators and so we can talk about zero being in the spectrum of $\widetilde{K}$.

PROPOSITION 15. *Zero is not in the spectrum of $\widetilde{K}$ if and only if $\pi_1(K)$ is big and $\chi(K) = 0$.*

Proof. Suppose that zero is not in the spectrum of $\widetilde{K}$. From the analog of Proposition 11, Γ must be big. Furthermore, from Properties 1 and 7, $\chi(K) = 0$.

Now suppose that $\pi_1(K)$ is big and $\chi(K) = 0$. From the analog of Proposition 11, $0 \notin \sigma(\triangle_0)$ and $0 \notin \sigma(\triangle_1)$. In particular, $\mathrm{Ker}(\triangle_0) = \mathrm{Ker}(\triangle_1) = 0$. From Properties 1 and 7, $\mathrm{Ker}(\triangle_2) = 0$. As $C^2(\widetilde{K}) = \mathrm{Ker}(\triangle_2) \oplus \overline{d_1 C^1(\widetilde{K})}$, we conclude that $0 \notin \sigma(\triangle_2)$. □

Let Γ be a finitely-presented group. Consider a fixed presentation of Γ consisting of g generators and r relations. Let K be the corresponding presentation complex. Then $\chi(K) = 1 - g + r$. Thus zero is not in the spectrum of $\widetilde{K}$ if and only if $\pi_1(K)$ is big and $g - r = 1$.

Recall that the *deficiency* $\mathrm{def}(\Gamma)$ is defined to be the maximum, over all finite presentations of Γ, of $g - r$. If $b_1^{(2)}(\Gamma) = 0$ then from the equation

$$\chi(K) = 1 - g + r = b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(K), \tag{5.6}$$

we obtain $\mathrm{def}(\Gamma) \leq 1$. This is the case, for example, when Γ is big or when Γ is amenable [5].

As any finite connected 2-dimensional CW-complex is homotopy-equivalent to a presentation complex, it follows from Proposition 15 that the answer to the zero-in-the-spectrum question is "yes" for universal covers of such complexes if and only if the following conjecture is true.

CONJECTURE 1. *If Γ is a big group then* $\mathrm{def}(\Gamma) \leq 0$.

REMARK. If $\pi_1(K)$ has property T then the ordinary first Betti number of K vanishes [6], and so $\chi(K) = 1 + b_2(K) > 0$. Thus zero lies in the spectrum of $\widetilde{K}$.

Now let Y be a 3-manifold satisfying the conditions of Proposition 13. If $\partial Y \neq \varnothing$, we define $\triangle_p$ on $\widetilde{Y}$ using absolute boundary conditions on $\partial \widetilde{Y}$.

PROPOSITION 16. *Zero lies in the spectrum of* $\widetilde{Y}$.

Proof. This is a consequence of Propositions 11 and 13. □

5.3 FOUR DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If M is a Riemannian 4-manifold then the Hodge decomposition gives

$$\begin{aligned}
\Lambda^0(M) &= \mathrm{Ker}(\triangle_0) \oplus \Lambda^0(M)/\,\mathrm{Ker}(d), \\
\Lambda^1(M) &= \mathrm{Ker}(\triangle_1) \oplus \overline{d\Lambda^0(M)} \oplus \Lambda^1(M)/\,\mathrm{Ker}(d), \\
\Lambda^2(M) &= \mathrm{Ker}(\triangle_2) \oplus \overline{d\Lambda^1(M)} \oplus \overline{*d\Lambda^1(M)}, \\
\Lambda^3(M) &= *\,\mathrm{Ker}(\triangle_1) \oplus \overline{*d\Lambda^0(M)} \oplus *\big(\Lambda^1(M)/\,\mathrm{Ker}(d)\big), \\
\Lambda^4(M) &= *\,\mathrm{Ker}(\triangle_0) \oplus *\big(\Lambda^0(M)/\,\mathrm{Ker}(d)\big).
\end{aligned} \tag{5.7}$$

Thus for the zero-in-the-spectrum question, it is enough to consider $\mathrm{Ker}(\triangle_0)$, $\mathrm{Ker}(\triangle_1)$, $\sigma\big(\triangle_0 \text{ on } \Lambda^0/\,\mathrm{Ker}(d)\big)$, $\sigma\big(\triangle_1 \text{ on } \Lambda^1/\,\mathrm{Ker}(d)\big)$ and $\mathrm{Ker}(\triangle_2)$.

Let Γ be a finitely-presented group. Recall that Γ is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of Γ. Embed the resulting presentation complex in $\mathbf{R}^5$ and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds X with fundamental group Γ. Given X, we have $\chi(X \# \mathbf{C}P^2) = \chi(X) + 1$. Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

$$\begin{aligned}
\{\chi(X) :\ & X \text{ is a closed connected oriented 4-manifold with} \\
& \pi_1(X) = \Gamma\} = \{n \in \mathbf{Z} : n \geq q(\Gamma)\}
\end{aligned} \tag{5.8}$$

for some $q(\Gamma)$. *A priori* $q(\Gamma) \in \mathbf{Z} \cup \{-\infty\}$, but in fact $q(\Gamma) \in \mathbf{Z}$ [17, Theorem 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of $q(\Gamma)$.

Suppose that $\pi_1(X) = \Gamma$. From Properties 4, 7 and 8 above,

$$\chi(X) = 2b_0^{(2)}(\Gamma) - 2b_1^{(2)}(\Gamma) + b_2^{(2)}(X). \tag{5.9}$$

In particular, if $b_1^{(2)}(\Gamma) = 0$ then $\chi(X) \geq 0$ and so $q(\Gamma) \geq 0$. This is the case, for example, when Γ is big or when Γ is amenable [5].