Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE ZERO-IN-THE-SPECTRUM QUESTION
Autor: LOTT, John
Kapitel: 5.3 Four Dimensions
DOI: https://doi.org/10.5169/seals-87882

Now let Y be a 3-manifold satisfying the conditions of Proposition 13. If $\partial Y \neq \varnothing$, we define $\triangle_p$ on $\widetilde{Y}$ using absolute boundary conditions on $\partial \widetilde{Y}$.

PROPOSITION 16. *Zero lies in the spectrum of* $\widetilde{Y}$.

Proof. This is a consequence of Propositions 11 and 13. □

5.3 Four Dimensions

In this subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If M is a Riemannian 4-manifold then the Hodge decomposition gives

$$\begin{aligned}
\Lambda^0(M) &= \operatorname{Ker}(\triangle_0) \oplus \Lambda^0(M)/\operatorname{Ker}(d), \\
\Lambda^1(M) &= \operatorname{Ker}(\triangle_1) \oplus \overline{d\Lambda^0(M)} \oplus \Lambda^1(M)/\operatorname{Ker}(d), \\
\Lambda^2(M) &= \operatorname{Ker}(\triangle_2) \oplus \overline{d\Lambda^1(M)} \oplus \overline{*d\Lambda^1(M)}, \\
\Lambda^3(M) &= *\operatorname{Ker}(\triangle_1) \oplus \overline{*d\Lambda^0(M)} \oplus *\big(\Lambda^1(M)/\operatorname{Ker}(d)\big), \\
\Lambda^4(M) &= *\operatorname{Ker}(\triangle_0) \oplus *\big(\Lambda^0(M)/\operatorname{Ker}(d)\big).
\end{aligned} \tag{5.7}$$

Thus for the zero-in-the-spectrum question, it is enough to consider $\operatorname{Ker}(\triangle_0)$, $\operatorname{Ker}(\triangle_1)$, $\sigma\big(\triangle_0 \text{ on } \Lambda^0/\operatorname{Ker}(d)\big)$, $\sigma\big(\triangle_1 \text{ on } \Lambda^1/\operatorname{Ker}(d)\big)$ and $\operatorname{Ker}(\triangle_2)$.

Let Γ be a finitely-presented group. Recall that Γ is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of Γ. Embed the resulting presentation complex in $\mathbf{R}^5$ and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds X with fundamental group Γ. Given X, we have $\chi(X \# \mathbf{C}P^2) = \chi(X) + 1$. Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

$$\begin{aligned}
&\{\chi(X) : X \text{ is a closed connected oriented 4-manifold with} \\
&\qquad \pi_1(X) = \Gamma\} = \{n \in \mathbf{Z} : n \geq q(\Gamma)\}
\end{aligned} \tag{5.8}$$

for some $q(\Gamma)$. *A priori* $q(\Gamma) \in \mathbf{Z} \cup \{-\infty\}$, but in fact $q(\Gamma) \in \mathbf{Z}$ [17, Theorem 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of $q(\Gamma)$.

Suppose that $\pi_1(X) = \Gamma$. From Properties 4, 7 and 8 above,

$$\chi(X) = 2b_0^{(2)}(\Gamma) - 2b_1^{(2)}(\Gamma) + b_2^{(2)}(X). \tag{5.9}$$

In particular, if $b_1^{(2)}(\Gamma) = 0$ then $\chi(X) \geq 0$ and so $q(\Gamma) \geq 0$. This is the case, for example, when Γ is big or when Γ is amenable [5].

PROPOSITION 17. *Let X be a closed 4-manifold. Then zero is not in the spectrum of $\widetilde{X}$ if and only if $\pi_1(X)$ is big and $\chi(X) = 0$.*

Proof. Suppose that zero is not in the spectrum of $\widetilde{X}$. Then from Proposition 11, $\pi_1(X)$ must be big. Furthermore, $\mathrm{Ker}(\triangle_2) = 0$. From Property 1 and (5.9), $\chi(X) = 0$.

Now suppose that $\pi_1(X)$ is big and $\chi(X) = 0$. From Proposition 11, $0 \notin \sigma(\triangle_0)$ and $0 \notin \sigma(\triangle_1)$. From Property 1 and (5.9), $\mathrm{Ker}(\triangle_2) = 0$. Then from (5.7), zero is not in the spectrum of $\widetilde{X}$. □

REMARK. If zero is not in the spectrum of $\widetilde{X}$ then it follows from Property 9 that in addition, $\tau(X) = 0$. Also, as will be explained later in Corollary 4, if $\pi_1(X)$ satisfies the Strong Novikov Conjecture then $\nu_*([X])$ vanishes in $\mathrm{H}_4(B\pi_1(X); \mathbf{C})$.

In summary, we have shown that the answer to the zero-in-the-spectrum question is "yes" for universal covers of closed 4-manifolds if and only if the following conjecture is true.

CONJECTURE 2. *If Γ is a big group then $q(\Gamma) > 0$.*

We now give some partial positive results on the zero-in-the-spectrum question for universal covers of closed 4-manifolds. Recall that there is a notion, due to Thurston, of a manifold having a geometric structure. This is especially important for 3-manifolds. The 4-manifolds with geometric structures have been studied by Wall [32].

PROPOSITION 18. *Let X be a closed 4-manifold. Then zero is in the spectrum of $\widetilde{X}$ if*

1. *$\pi_1(X)$ has property T or*
2. *X has a geometric structure (and an arbitrary Riemannian metric) or*
3. *X has a complex structure (and an arbitrary Riemannian metric).*

Proof.

1. If X has property T then the ordinary first Betti number of X vanishes [6]. Thus $\chi(X) = 2 + b_2(X) > 0$. Part 1. of the proposition follows.
2. The geometries of [32] all fall into at least one of the following classes:

a. $b_0^{(2)} \neq 0 : S^4, S^2 \times S^2, \mathbf{C}P^2$.

b. $0 \in \sigma\big(\triangle_0$ on $\Lambda^0/\operatorname{Ker}(d)\big) : \mathbf{R}^4, S^3 \times \mathbf{R}, S^2 \times \mathbf{R}^2, Nil^3 \times \mathbf{R}, Nil^4, Sol_0^4, Sol_1^4, Sol_{m,n}^4$.

c. $b_1^{(2)} \neq 0 : S^2 \times H^2$.

d. $0 \in \sigma\big(\triangle_1$ on $\Lambda^1/\operatorname{Ker}(d)\big) : H^3 \times \mathbf{R}, \widetilde{SL_2} \times \mathbf{R}, H^2 \times \mathbf{R}^2$.

e. $\chi > 0 : H^4, H^2 \times H^2, \mathbf{C}H^2$.

Part 2. of the proposition follows.

3. Suppose that zero is not in the spectrum of $\widetilde{X}$. From Properties 7 and 9, $\chi(X) = \tau(X) = 0$. From the classification of complex surfaces, X has a geometric structure [32, p. 148–149]. This contradicts part 2. of the proposition. □

5.4 More Dimensions

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [18]. Let X be a closed Riemannian manifold. If $\dim(X)$ is even, consider the operator $d + d^*$ on $\Lambda^*(X)$. Give $\Lambda^*(X)$ the $\mathbf{Z}_2$-grading coming from (3.12). Then the signature $\tau(X)$ equals the index of $d+d^*$. To say this in a more complicated way, the operator $d+d^*$ defines a element $[d+d^*]$ of the K-homology group $K_0(X)$. Let $\eta : X \to \text{pt.}$ be the (only) map from X to a point. Then $\eta_*([d+d^*]) \in K_0(\text{pt.})$. There is a map $A : K_0(\text{pt.}) \to K_0(\mathbf{C})$ which is the identity, as both sides are $\mathbf{Z}$. So we can say that $\tau(X) = A\big(\eta\big([d + d^*]\big)\big) \in K_0(\mathbf{C})$.

We now extend the preceding remarks to the case of a group action. Let M be a normal cover of X with covering group Γ. The fiber bundle $M \to X$ is classified by a map $\nu : X \to B\Gamma$, defined up to homotopy. Let $\widetilde{d}$ be exterior differentiation on M. Consider the operator $\widetilde{d} + \widetilde{d}^*$. Taking into account the action of Γ on M, one can define a refined index $\operatorname{ind}(\widetilde{d} + \widetilde{d}^*) \in K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^*-algebra of Γ.

We recall the statement of the Strong Novikov Conjecture (SNC) [18, 29]. This is a conjecture about a countable discrete group Γ, namely that the assembly map $A : K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.