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semidirect product of K and a connected simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G [12, Theorem 6.4]. We may as well assume that $X = \Gamma\backslash G/K$. By passing to a finite cover, we may assume that K is trivial. That is, X is a nilmanifold. From [27, Corollary 7.28], $\mathrm{H}^p(X;\mathbf{C}) \cong \mathrm{H}^p(g,\mathbf{C})$, the Lie algebra cohomology of g. From [7], $\mathrm{H}^p(g,\mathbf{C}) \neq 0$ for all $p \in [0,\dim(X)]$. Thus for all $p \in [0,\dim(X)]$, $\mathrm{H}^p(X;\mathbf{C}) \neq 0$.

Now let ω be a nonzero harmonic p-form on X. Let $\pi^*\omega$ be its pullback to $\widetilde{X}$. The idea is to construct low-energy square-integrable p-forms on X by multiplying $\pi^*\omega$ by appropriate functions on X. We define the functions as in [2, §2]. Take a smooth triangulation of X and choose a fundamental domain F for the lifted triangulation of $\widetilde{X}$. If E is a finite subset of $\pi_1(X)$, let χ_H be the characteristic function of $H = \bigcup_{g\in E} g \cdot F$. Given numbers $0 < \epsilon_1 < \epsilon_2 < 1$, choose a nonincreasing function $\psi \in C_0^\infty\big([0,\infty)\big)$ which is identically one on $[0,\epsilon_1]$ and identically zero on $[\epsilon_2,\infty)$. Define a compactly-supported function f_E on $\widetilde{X}$ by $f_E(m) = \psi\big(d(m,H)\big)$. Then there is a constant $C_1 > 0$, independent of E, such that

$$\int_{\widetilde{X}} |df_E|^2 \le C_1 \,\mathrm{area}(\partial H). \tag{5.12}$$

Define $\rho_E \in \Lambda^p(\widetilde{X})$ by $\rho_E = f_E \cdot \pi^*\omega$. We have $d\rho_E = df_E \wedge \pi^*\omega$ and $d^*\rho_E = -i(df_E)\,\pi^*\omega$. As f_E is identically one on H, it follows that there is a constant $C > 0$, independent of E, such that

$$\frac{\int_{\widetilde{X}} \left[|d\rho_E|^2 + |d^*\rho_E|^2\right]}{\int_{\widetilde{X}} |\rho_E|^2} \le C\frac{\mathrm{area}(\partial H)}{\mathrm{vol}(H)}. \tag{5.13}$$

As $\pi_1(X)$ is amenable, by an appropriate choice of E this ratio can be made arbitrarily small. Thus $0 \in \sigma(\triangle_p)$. □

QUESTION. Does the conclusion of Proposition 20 hold if we only assume that $\pi_1(X)$ is amenable ?

6. Topologically Tame Manifolds

Another class of manifolds for which one can hope to get some nontrivial results about the zero-in-the-spectrum question is given by *topologically tame* manifolds, meaning manifolds M which are diffeomorphic to the interior of a compact manifold N with boundary. If M has finite volume then $\mathrm{Ker}(\triangle_0) \neq 0$, so we restrict our attention to the infinite volume case. A limited result is given by Corollary 2.

An interesting class of topologically tame manifolds consists of those which are radially symmetric. This means that M is diffeomorphic to $\mathbf{R}^n$, with a metric which is given on $\mathbf{R}^n - \{0\} \cong (0,\infty) \times S^{n-1}$ by

$$g = dr^2 + \phi^2(r) d\Omega^2. \tag{6.1}$$

Here $d\Omega^2$ is the standard metric on S^{n-1}, $r \in (0,\infty)$, $\phi \in C^\infty([0,\infty))$, $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(r) > 0$ for $r > 0$.

PROPOSITION 21. *Suppose that there is a constant* $c \geq 0$ *such that* $\mathrm{Ricci}_M \geq -c^2$. *Then* $0 \in \sigma(\triangle_p)$ *for some* p.

Proof. We may assume that $\mathrm{vol}(M) = \infty$. Suppose first that

$$\liminf_{r\to\infty} \phi(r) < \infty .$$

Then there is a constant $C > 0$ and a sequence $\{r_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} r_j = \infty$ and $\phi(r_j) \leq C$. Let D_j be the domain in M defined by $r \leq r_j$. Then $\mathrm{area}(D_j) \leq C^{n-1}\,\mathrm{vol}(S^{n-1})$ and $\lim_{j\to\infty} \mathrm{vol}(D_j) = \infty$. Thus M is not open at infinity. By Proposition 6, $0 \in \sigma(\triangle_0)$.

Now suppose that $\liminf_{r\to\infty} \phi(r) = \infty$. We want to show that M is hyperEuclidean and apply Proposition 7. Consider a map $F : M \to \mathbf{R}^n$ given in polar coordinates by

$$F(r,\theta) = (s(r),\theta)\,, \tag{6.2}$$

for some $s : [0,\infty) \to [0,\infty)$. The condition for F to be distance-nonincreasing is

$$|s'(r)| \leq 1, \qquad s(r) \leq \phi(r)\,. \tag{6.3}$$

If $\lim_{r\to\infty} s(r) = \infty$ then F is a proper map of degree one. It remains to construct s satisfying (6.3).

Put

$$\widetilde{\phi}(r) = \inf_{v\in[r,\infty)} \phi(v)\,. \tag{6.4}$$

Replacing ϕ by $\widetilde{\phi}$, we may assume that ϕ is monotonically nondecreasing. Thinking of $\phi(r)$ as representing the trajectory of a car in front of us which is blocking the road, with our car's velocity bounded above by one, it is intuitively clear that we can find a trajectory $s(r)$ for our car such that

$\lim_{r\to\infty} s(r) = \infty$. More precisely, let $\rho \in C^\infty([0,2])$ be a nondecreasing function which is identically zero near 0, identically one near 2 and satisfies $\rho'(x) \leq 1$ for all $x \in [0,2]$. Put $r_0 = 0$ and define $\{r'_j\}_{j=0}^\infty$ and $\{r_j\}_{j=1}^\infty$ inductively by

$$\begin{aligned} r'_j &= \inf\{r : r \geq r_j + 2 \text{ and } \phi(r) \geq j+1\}, \\ r_{j+1} &= r'_j + 2. \end{aligned} \tag{6.5}$$

Define s by

$$s(r) = \begin{cases} j & \text{if } r \in [r_j, r'_j] \\ j + \rho(r - r'_j) & \text{if } r \in [r'_j, r_{j+1}]. \end{cases} \tag{6.6}$$

Then s satisfies (6.3) and $\lim_{r\to\infty} s(r) = \infty$. □

QUESTION. What can one say in the radially symmetric case without the assumption of a lower bound on the Ricci curvature?

Another interesting class of topologically tame manifolds consists of those which are hyperbolic, that is, of constant sectional curvature -1. Complete hyperbolic manifolds are divided into those which are *geometrically finite* and those which are *geometrically infinite*. Roughly speaking, M is geometrically finite if its set of ends consists of a finite number of standard cusps and flares.

PROPOSITION 22 (Mazzeo-Phillips [23, Theorem 1.11]). *Let M be an infinite-volume geometrically finite hyperbolic manifold. If* $\dim(M) = 2k$ *then* $\dim(\mathrm{Ker}(\triangle_k)) = \infty$. *If* $\dim(M) = 2k+1$ *then* $\sigma(\triangle_k) = \sigma(\triangle_{k+1}) = [0,\infty)$.

The paper [23] also computes $\dim(\mathrm{Ker}(\triangle_p))$ for such manifolds.

In general, geometrically infinite hyperbolic manifolds can have wild end behavior. However, in three dimensions one can show that the ends have a fairly nice structure. This is used to prove the next result.

PROPOSITION 23 (Canary [4, Theorem A]). *If M is a geometrically infinite topologically tame hyperbolic* 3*-manifold then* $0 \in \sigma(\triangle_0)$.

Proof. The method of proof is to show that M is not open at infinity and then apply Theorem 6. See [4] for details. □

Thus zero lies in the spectrum of all topologically tame hyperbolic 3-manifolds. From Proposition 2, the same statement is true for compactly-supported modifications of such manifolds.

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