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> the contrary conclusion: my consciousness is focused on the successive images, or more exactly, on the global image; the arguments themselves wait, so to speak, in the antechamber to be introduced at the beginning of the "precising" phase. *(Hadamard, 1945, 80-81)*

Students who have little of this internal structure see in a proof just a sequence of steps which they feel forced to commit to memory for an examination:

> Maths courses, having a habit of losing every student by the end of the first lecture, definitely create a certain amount of negative feeling (as well as a considerable amount of apathy) and the aim for the exam becomes the anti-goal of 'aiming to get through so I don't have to retake' rather than the goal of 'working hard to do well because I enjoy the subject'. *(Female mathematics undergraduate, 2nd Year)*

This use of memory for routinizing sequential procedures is a valuable human tool when the mental objects to be manipulated will not all fit in the focus of attention at the same time. The memory scratch-pad available is small — about 7 ± 2 items according to Miller (1956).

When individuals fail to perform the compression satisfactorily they do not have mental objects which can be held simultaneously in memory (Linchevski & Sfard, 1991). They are then forced into using method (3) as a *defence* mechanism — remembering routine procedures and internalising them so that they need less conscious memory to process. The problem is that such procedures can only be performed in time one after another, leading to an inflexible *procedural* view of mathematics. Such procedural learning may work at one level in routine examples, but it produces an escalating degree of difficulty at successive stages because it is more difficult to co-ordinate processes than manipulate concepts. *The failing student fails because he or she is doing a different kind of mathematics which is harder than the flexible thinking of the successful mathematician.*

THE TRANSITION TO FORMAL MATHEMATICS

Students usually find formal mathematics in conflict with their experience. It is no longer about procepts — symbols representing a process to be computed or manipulated to give a result. The concepts in formal mathematics are no longer related so directly to objects in the real world. Instead the mathematics has been systematised (à la Bourbaki) and presented as a polished theory in which mathematical concepts are *defined* as mental objects having certain minimal fundamental properties and all other properties are *deduced* from this. The definitions are often complex linguistic statements involving several quantifiers.

This formal meaning is difficult to attain. For instance, of a group of mathematics education students studying analysis as "an essential part of their education", *none* could give the definition of the convergence of a sequence after two weeks of using the idea in lectures. Of course these students are not the "best" students studying analysis, but their failure is typical of a spectrum of levels of failure in understanding mathematical analysis. Even distinguished mathematicians remember their struggles with the subject:

> ... I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother me. I could juggle them quite well... [but] I was stumped by the infinitesimal subtlety of epsilonic analysis. I could read analytic proofs, remember them if I made an effort, and reproduce them, sort of, but I didn't really know what was going on.
>
> *(Halmos, 1985, p. 47)*

Halmos was fortunate enough to eventually find out what the 'real knowing' was all about:

> ... one afternoon something happened. I remember standing at the blackboard in Room 213 of the mathematics building talking with Warren Ambrose and suddenly I understood epsilon. I understood what limits were, and all of that stuff that people were drilling in me became clear. I sat down that afternoon with the calculus textbook by Granville, Smith, and Longley. All of that stuff that previously had not made any sense became obvious...
>
> *(Halmos in Albers & Alexanderson, 1985, p. 123)*

Regrettably many students never reach enlightenment. Although visual images may suggest theorems, the use of definitions demands a new form of compression of knowledge. The definitions used in mathematics must be *written* so that the information may be scanned to allow different parts to become the focus of attention at different levels. For instance, the definition of continuity is heard as:

> For any *ay* in the domain of the function *eff*, given an epsilon greater than zero, there exists a delta greater than zero such that if *ex* lies in the domain of *eff* and the absolute value of *ex* minus *ay* is less than delta then the absolute value of *eff of ex* minus *eff of ay* is less than epsilon.

It is far too long to be held meaningfully in the focus of attention through hearing alone. It only begins to make sense when compressed in symbolic writing concentrating first on continuity at a point $a \in D$:

A function $f\colon D \to \mathbf{R}$ is continuous at $a \in D$ if:

$\forall \varepsilon > 0,\ \exists \delta > 0$ such that $x \in D,\ |x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

Then various parts can be scanned and chunked together:

$$\forall \varepsilon > 0, \exists \delta > 0$$

$$\text{such that } \boxed{\boxed{x \in D, |x-a| < \delta} \text{ implies } \boxed{|f(x) - f(a)| < \varepsilon}}\ .$$

This may be focused at one level as

For all $\varepsilon > 0$, there is a $\delta > 0$ such that $\boxed{\text{an implication is satisfied}}$,

or at another as

For all $\varepsilon > 0$, there is a $\delta > 0$

$$\text{such that } \boxed{\boxed{\text{one condition}} \text{ implies } \boxed{\text{another}}}\ .$$

It is possible to concentrate on *part* of the sentence, such as

$$\boxed{x \in D, |x-a| < \delta}\ ,$$

to interpret what it means, in this case, "x lies in D (which means that $f(x)$ is defined) and the distance between x and a is less than δ." In this way we may use the written word to scan the information linearly or non-linearly, or focus on important chunks of information to build up the conceptual structure and relationships between the parts.

In a pilot study I interviewed mathematics majors at a university with a high reputation for pure mathematics, and found a wide difference in performance between the unsuccessful for whom the theory made no sense at all and the successful who understood the logical necessity of proof. *But even the most able student interviewed did not always internalise the definition and operate with its full meaning several weeks after it had been given and used continually in the lectures.* Others who were failing to use the definition went back to their visual images of a continuous function as a "graph drawn without taking the pencil off the paper" and performed *thought experiments* based on these images. They considered the statement of the intermediate value theorem to be simple and "obvious" but found the formal proof impossible to follow. Students such as these resort to damage limitation using rote-learning of procedures as reported in another investigation:

> ... everyone is faced with courses whose purpose they have failed to grasp, let alone their finer details. Faced with this problem, most people set about finding typical questions and memorising the typical answers. Many gain excellent marks in courses of which they have no knowledge. *(Second year university mathematics student)*

What else can the failing student do? As Freudenthal said succinctly:

> ... the only thing the pupil can do with the ready-made mathematics which he is offered is to reproduce it. *(Freudenthal, 1973, p. 117)*

CAN WE TEACH STUDENTS TO "THINK MATHEMATICALLY"?

Can we encourage students to think like mathematicians? Even though we may not make every student a budding research mathematician, can we not alter attitudes and methods of doing mathematics that fosters a *creative* way of learning?

If students are given a suitable environment to relax and think about problems of an appropriate level, then such aspirations prove to be easy to attain. Typical problems (to be found in *Thinking Mathematically*, Mason *et al.,* 1982) include:

- If a square is cut into regions by straight lines, how many colours are needed so that no two adjoining regions are painted the same colour?
- Into how many squares can one cut a square?

These problems, on the face of it fairly easy, prove to be challenging, especially when *proof* is required — for instance proving that it is *not* possible to cut a square into two, three, or five squares. The latter statement proves to be true under certain circumstances, but false under others. I will not spoil it by refining the conditions on the problem, except to say that the alternative solution was given by a thirteen year old girl in a master class, when it had not occurred to me or to several hundred mathematics undergraduates over a decade of problem-solving classes.

Reflective thinking in mathematics is built up by Mason *et al.,* following the *How To Solve It* approach of Pólya (1945), but made more student-friendly by breaking problem-solving into three phases. The first is an *entry* phase in which the student must focus on the nature of the problem by asking "what do I want", reflect on any knowledge that may be available to begin the attack