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In particular, it follows that $J(\omega)$, the elliptic modular function [5], is real at ω. In each case one has to determine the possible reflections ρ, determine their fixed-point sets, and add a suitable τ_1.

We consider the rectangular case (1) of the lemma, for application in the next section. Let

(5.19) $$\omega_1 = 1, \omega_2 = \omega = i\omega'', \omega'' > 1$$

be a normalized basis. For $a = 1$, l_α is the real axis, $\omega_0 = 0$, or $\omega_0 = 1$, $b = ib_2$, or $b = \frac{1}{2} + ib_2$, $0 \leqslant b_2 < \omega''$. In the first case $\omega_0' = 0$, or $\omega_0' = \omega$, while there is no ω_0' in the second case. Thus, we have

(5.20) $$\rho(t) = \bar{t} + ib_2, \; FP(\rho) = \{Im\, t = b_2/2\} \cup \{Im\, t = (b_2 + \omega'')/2\}.$$

For $a = -1$, l_α is the imaginary axis, $\omega_0 = 0$ or $\omega_0 = \omega$, $b = b_1$, or $b = b_1 + i\omega''/2$, $0 \leqslant b_1 < 1$. $\omega_0' = 0, 1$ in the first case, and there is no ω_0' in the second case. We have

(5.21) $$\rho(t) = -\bar{t} + b_1, \; FP(\rho) = \{Re\, t = b_1/2\} \cup \{Re\, t = (b_1 + 1)/2\}.$$

If $\varepsilon_1 = -1$, then

(5.22) $$FP(\tau_1) = \{c_1/2, (c_1 + \omega_1)/2, (c_1 + \omega_2)/2, (c_1 + \omega_1 + \omega_2)/2\}.$$

If we have $\varepsilon_1 = +1$, $2c_1 \in \Lambda$, $c_1 \notin \Lambda$, then τ_1 has no fixed points. τ_1 is then the deck transformation of an unbranched covering of another torus.

6. Embedding of tori

We turn to the problem of concretely realizing the data of the previous section in the main case. Given a complex torus $\Gamma = \mathbf{C}/\Lambda$, with a pair of holomorphic involutions induced by

(6.1) $$\tau_i(t) = -t + c_i, \; i = 1, 2,$$

we look for a pair of two-fold branched coverings

(6.2) $$\pi_i : \Gamma \to \mathbf{P}_1, \; \pi_i \circ \tau_i = \pi_i, \; i = 1, 2.$$

The problem is immediately solved by taking

(6.3) $$z_i = \pi_i(t) \equiv \wp(t - c_i/2), \; i = 1, 2,$$

where

$$\mathscr{P}(t) = \frac{1}{t^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(t-\omega)^2} - \frac{1}{\omega^2} \right) \tag{6.4}$$

is the Weierstrass $\mathscr{P}$-function [5], [6]. We set

$$\pi(t) = (\pi_1(t), \pi_2(t)) \, . \tag{6.5}$$

If $\pi(s_0) = \pi(t_0)$, $s_0 \neq t_0$, then $s_0 \equiv -t_0 + c_i$, mod Λ. Thus π will be one-to-one, as a map into $\mathbf{P}_1 \times \mathbf{P}_1$, if we assume

$$c_2 - c_1 \notin \Lambda \, . \tag{6.6}$$

To represent π as a map into $\mathbf{P}_2$ with homogeneous coordinates $\zeta, z_1 = \zeta_1/\zeta_0, z_2 = \zeta_2/\zeta_0$, we again use the sigma function (4.13). We have [6]

$$\mathscr{P}(t) = -\partial_t^2 \log S(t) = -\frac{\Delta}{S(t)^2} \, , \; \Delta = S(t)S''(t) - S'(t)^2 \, . \tag{6.7}$$

Since $\Delta(0) = -S'(0)^2 \neq 0$, we may write π as

$$\begin{aligned} \zeta_0 &= S(t - c_1/2)^2 S(t - c_2/2)^2 \\ \zeta_1 &= \Delta(t - c_1/2) S(t - c_2/2)^2 \\ \zeta_2 &= \Delta(t - c_2/2) S(t - c_1/2)^2 \end{aligned} \tag{6.8}$$

The branch points of the map π_1 are given by (5.22), with $\omega_1 = 1$, and $\omega_2 = \omega$. By (6.6) the curve π has no finite singular points. Since $\pi_i(t)$ has a pole of order two at $t = c_i/2$, $i = 1, 2$; the plane curve has two cusps on the line at ∞ corresponding to these two parameter values. Such curves are considered in [3], for example.

To find the equation $G(z_1, z_2) = 0$ of this plane curve, we change the variable, $t \to t - c_1/2$, so that $G(\mathscr{P}(t), \mathscr{P}(t + c)) = 0$, where

$$c = (c_1 - c_2)/2 \, . \tag{6.9}$$

We set

$$\begin{gathered} x = \mathscr{P}(t + c), p = \mathscr{P}(t), p' = \mathscr{P}'(t), \\ \beta = \mathscr{P}(c), \beta' = \mathscr{P}'(c) \, . \end{gathered} \tag{6.10}$$

The addition theorem and differential equation satisfied by $\mathscr{P}$ [6] give

$$x + p + \beta = \frac{1}{4} \left(\frac{p' - \beta'}{p - \beta} \right)^2 , \; p'^2 = 4p^3 - g_2 p - g_3 \, .$$

We rewrite these as

$$(p' - \beta')^2 = A(x,p),\ p'^2 = B(p)\ ,$$

and eliminate p'. This gives

$$F(x,p) \equiv F(x,p,\beta,\beta') \equiv (A - B - \beta'^2)^2 - 4\beta'^2 B = 0\ . \tag{6.11}$$

Note that $A - B$ is quadratic in p, and $\beta'^2 = B(\beta)$. Since F is an even function of β', and $\mathscr{P}$ is an even function, changing c to $-c$ shows that we also have $F(p,x) = 0$. Since the coefficient of x^2 in F is $16(p - \beta)^2$, we must have

$$F(x,p) = G(x,p)\,(p-\beta)^2\ .$$

Expanding in powers of $p - \beta$ gives

$$F(x,\beta) = 0,\ \partial_p F(x,\beta) = 0\ ,$$

$$\begin{aligned} G(x,p) = {} & (1/2)\,\partial_p^2 F(x,\beta) + (1/6)\,\partial_p^3 F(x,\beta)\,(p-\beta) \\ & + (1/24)\,\partial_p^4 F(x,\beta)\,(p-\beta)^2\ . \end{aligned} \tag{6.12}$$

After some computation we get

$$\begin{aligned} G(z_1,z_2) = {} & (z_1-\beta)^2(z_2-\beta)^2 + \beta_1(z_1-\beta)\,(z_2-\beta) \\ & + \beta_2(z_1+z_2-2\beta) + \beta_3\ , \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} \beta_1 = {} & -(12\beta^2 - g_2)/2,\ \beta_2 = -B(\beta), \\ \beta_3 = {} & (12\beta^2 - g_2)^2 - 3\beta B(\beta)\ . \end{aligned} \tag{6.14}$$

Next we consider the reality condition (3.11). From (5.9) and (6.4) we get

$$\overline{\mathscr{P}(t)} = a^2\,\mathscr{P}(a\bar{t})\ . \tag{6.15}$$

By definition $g_2 = 60G_2,\ g_3 = 140G_3$, where [6]

$$G_k = \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^{2k}}\ .$$

It follows from (5.9) that $\bar{G}_k = a^{2k}G_k$, so that

$$\bar{g}_2 = a^4 g_2,\ \bar{g}_3 = a^6 g_3\ . \tag{6.16}$$

By (5.7) we have $c = (c_1 - a\bar{c}_1 - b + a\bar{b})/2$, so that $a\bar{c} = -c$. Hence,

$$\bar{\beta} = a^2\beta,\ \bar{\beta}_1 = a^4\beta_1,\ \bar{\beta}_2 = a^6\beta_2,\ \bar{\beta}_3 = a^8\beta_3\ . \tag{6.17}$$

To satisfy (3.11) we *redefine*

$$\pi_i(t) = a\mathscr{P}(t - c_i/2) , \tag{6.18}$$

and set

$$G_0(z_1, z_2) = a^4 G(z_1/a, z_2/a) , \tag{6.19}$$

so that

$$\overline{G_0(z_1, z_2)} = G_0(\bar{z}_2, \bar{z}_1) . \tag{6.20}$$

In summary we have

PROPOSITION 6.1. *Let* $\Lambda = \mathbf{C}/\Lambda$ *have the holomorphic involutions (6.1) intertwined by the anti-holomorphic involution (5.6). Then* (Γ, ρ, τ_i) *is realized by the map (6.5), (6.18) onto the quartic curve* $G_0(z_1, z_2) = 0$ *given by (6.13), (6.14), (6.19). If the fixed-point set of* ρ *is non-empty, then this is the complexification of the real curve* $G_0(z, \bar{z}) = 0$.

7. A RECTANGULAR LATTICE

We consider the special case of Λ, ρ, τ_i as given in (5.19), (5.6), (6.1), with

$$a = +1, b = 0, \bar{c}_2 = c_1 = c_1' + ic_1'', c = ic_1'' . \tag{7.1}$$

From (6.16), (6.15) it follows that g_2, g_3, β are real, and β' is purely imaginary. Thus, the coefficients $\beta_1, \beta_2, \beta_3$ of $G(z_1, z_2)$ are real. With $t = t' + it''$, we have

$$FP(\rho) = \{t'' = 0\} \cup \{t'' = \omega''/2\} , \tag{7.2}$$

$$\tau_1\{t'' = 0\} = \{t'' = c_1''\}, \ \tau_1\{t'' = \omega''/2\} = \{t'' = c_1'' + \omega''/2\} . \tag{7.3}$$

Let us assume that $0 < c_1'' < \omega''/2$. Then the torus Λ is divided into four annuli

$$A_1 = \{0 < t'' < c_1''\}, \ A_2 = \{c_1'' < t'' < \omega''/2\} ,$$
$$A_3 = \{\omega''/2 < t'' < c_1'' + \omega''/2\}, \ A_4 = \{c_1'' + \omega''/2 < t'' < \omega''\} .$$

The fixed points of τ_1 are, by (5.22),

$$c_1/2, (c_1 + 1)/2 \in A_1 , \tag{7.4}$$

$$(c_1 + i\omega'')/2, (c_1 + 1 + i\omega'')/2 \in A_3 . \tag{7.5}$$