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In particular, it follows that  $J(\omega)$ , the elliptic modular function [5], is real at  $\omega$ . In each case one has to determine the possible reflections  $\rho$ , determine their fixed-point sets, and add a suitable  $\tau_1$ .

We consider the rectangular case (1) of the lemma, for application in the next section. Let

$$(5.19) \quad \omega_1 = 1, \omega_2 = \omega = i\omega'', \omega'' > 1$$

be a normalized basis. For  $a = 1$ ,  $l_\alpha$  is the real axis,  $\omega_0 = 0$ , or  $\omega_0 = 1$ ,  $b = ib_2$ , or  $b = \frac{1}{2} + ib_2$ ,  $0 \leq b_2 < \omega''$ . In the first case  $\omega'_0 = 0$ , or  $\omega'_0 = \omega$ , while there is no  $\omega'_0$  in the second case. Thus, we have

$$(5.20) \quad \rho(t) = \bar{t} + ib_2, FP(\rho) = \{Im t = b_2/2\} \cup \{Im t = (b_2 + \omega'')/2\}.$$

For  $a = -1$ ,  $l_\alpha$  is the imaginary axis,  $\omega_0 = 0$  or  $\omega_0 = \omega$ ,  $b = b_1$ , or  $b = b_1 + i\omega''/2$ ,  $0 \leq b_1 < 1$ .  $\omega'_0 = 0, 1$  in the first case, and there is no  $\omega'_0$  in the second case. We have

$$(5.21) \quad \rho(t) = -\bar{t} + b_1, FP(\rho) = \{Re t = b_1/2\} \cup \{Re t = (b_1 + 1)/2\}.$$

If  $\varepsilon_1 = -1$ , then

$$(5.22) \quad FP(\tau_1) = \{c_1/2, (c_1 + \omega_1)/2, (c_1 + \omega_2)/2, (c_1 + \omega_1 + \omega_2)/2\}.$$

If we have  $\varepsilon_1 = +1$ ,  $2c_1 \in \Lambda$ ,  $c_1 \notin \Lambda$ , then  $\tau_1$  has no fixed points.  $\tau_1$  is then the deck transformation of an unbranched covering of another torus.

## 6. EMBEDDING OF TORI

We turn to the problem of concretely realizing the data of the previous section in the main case. Given a complex torus  $\Gamma = \mathbf{C}/\Lambda$ , with a pair of holomorphic involutions induced by

$$(6.1) \quad \tau_i(t) = -t + c_i, i = 1, 2,$$

we look for a pair of two-fold branched coverings

$$(6.2) \quad \pi_i: \Gamma \rightarrow \mathbf{P}_1, \pi_i \circ \tau_i = \pi_i, i = 1, 2.$$

The problem is immediately solved by taking

$$(6.3) \quad z_i = \pi_i(t) \equiv \mathcal{P}(t - c_i/2), i = 1, 2,$$

where

$$(6.4) \quad \mathcal{P}(t) = \frac{1}{t^2} + \sum_{\omega \in \Lambda - \{0\}} \left( \frac{1}{(t - \omega)^2} - \frac{1}{\omega^2} \right)$$

is the Weierstrass  $\mathcal{P}$ -function [5], [6]. We set

$$(6.5) \quad \pi(t) = (\pi_1(t), \pi_2(t)) .$$

If  $\pi(s_0) = \pi(t_0)$ ,  $s_0 \neq t_0$ , then  $s_0 \equiv -t_0 + c_i \pmod{\Lambda}$ . Thus  $\pi$  will be one-to-one, as a map into  $\mathbf{P}_1 \times \mathbf{P}_1$ , if we assume

$$(6.6) \quad c_2 - c_1 \notin \Lambda .$$

To represent  $\pi$  as a map into  $\mathbf{P}_2$  with homogeneous coordinates  $\zeta, z_1 = \zeta_1/\zeta_0, z_2 = \zeta_2/\zeta_0$ , we again use the sigma function (4.13). We have [6]

$$(6.7) \quad \mathcal{P}(t) = -\partial_t^2 \log S(t) = -\frac{\Delta}{S(t)^2}, \quad \Delta = S(t)S''(t) - S'(t)^2 .$$

Since  $\Delta(0) = -S'(0)^2 \neq 0$ , we may write  $\pi$  as

$$(6.8) \quad \begin{aligned} \zeta_0 &= S(t - c_1/2)^2 S(t - c_2/2)^2 \\ \zeta_1 &= \Delta(t - c_1/2) S(t - c_2/2)^2 \\ \zeta_2 &= \Delta(t - c_2/2) S(t - c_1/2)^2 \end{aligned}$$

The branch points of the map  $\pi_1$  are given by (5.22), with  $\omega_1 = 1$ , and  $\omega_2 = \omega$ . By (6.6) the curve  $\pi$  has no finite singular points. Since  $\pi_i(t)$  has a pole of order two at  $t = c_i/2$ ,  $i = 1, 2$ ; the plane curve has two cusps on the line at  $\infty$  corresponding to these two parameter values. Such curves are considered in [3], for example.

To find the equation  $G(z_1, z_2) = 0$  of this plane curve, we change the variable,  $t \rightarrow t - c_1/2$ , so that  $G(\mathcal{P}(t), \mathcal{P}(t + c)) = 0$ , where

$$(6.9) \quad c = (c_1 - c_2)/2 .$$

We set

$$(6.10) \quad \begin{aligned} x &= \mathcal{P}(t + c), p = \mathcal{P}(t), p' = \mathcal{P}'(t), \\ \beta &= \mathcal{P}(c), \beta' = \mathcal{P}'(c) . \end{aligned}$$

The addition theorem and differential equation satisfied by  $\mathcal{P}$  [6] give

$$x + p + \beta = \frac{1}{4} \left( \frac{p' - \beta'}{p - \beta} \right)^2, \quad p'^2 = 4p^3 - g_2p - g_3 .$$

We rewrite these as

$$(p' - \beta')^2 = A(x, p), p'^2 = B(p),$$

and eliminate  $p'$ . This gives

$$(6.11) \quad F(x, p) \equiv F(x, p, \beta, \beta') \equiv (A - B - \beta'^2)^2 - 4\beta'^2 B = 0.$$

Note that  $A - B$  is quadratic in  $p$ , and  $\beta'^2 = B(\beta)$ . Since  $F$  is an even function of  $\beta'$ , and  $\mathcal{P}$  is an even function, changing  $c$  to  $-c$  shows that we also have  $F(p, x) = 0$ . Since the coefficient of  $x^2$  in  $F$  is  $16(p - \beta)^2$ , we must have

$$F(x, p) = G(x, p) (p - \beta)^2.$$

Expanding in powers of  $p - \beta$  gives

$$(6.12) \quad \begin{aligned} F(x, \beta) &= 0, \partial_p F(x, \beta) = 0, \\ G(x, p) &= (1/2) \partial_p^2 F(x, \beta) + (1/6) \partial_p^3 F(x, \beta) (p - \beta) \\ &\quad + (1/24) \partial_p^4 F(x, \beta) (p - \beta)^2. \end{aligned}$$

After some computation we get

$$(6.13) \quad \begin{aligned} G(z_1, z_2) &= (z_1 - \beta)^2 (z_2 - \beta)^2 + \beta_1 (z_1 - \beta) (z_2 - \beta) \\ &\quad + \beta_2 (z_1 + z_2 - 2\beta) + \beta_3, \end{aligned}$$

where

$$(6.14) \quad \begin{aligned} \beta_1 &= -(12\beta^2 - g_2)/2, \beta_2 = -B(\beta), \\ \beta_3 &= (12\beta^2 - g_2)^2 - 3\beta B(\beta). \end{aligned}$$

Next we consider the reality condition (3.11). From (5.9) and (6.4) we get

$$(6.15) \quad \overline{\mathcal{P}(t)} = a^2 \mathcal{P}(at).$$

By definition  $g_2 = 60G_2, g_3 = 140G_3$ , where [6]

$$G_k = \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^{2k}}.$$

It follows from (5.9) that  $\bar{G}_k = a^{2k} G_k$ , so that

$$(6.16) \quad \bar{g}_2 = a^4 g_2, \bar{g}_3 = a^6 g_3.$$

By (5.7) we have  $c = (c_1 - a\bar{c}_1 - b + a\bar{b})/2$ , so that  $a\bar{c} = -c$ . Hence,

$$(6.17) \quad \bar{\beta} = a^2 \beta, \bar{\beta}_1 = a^4 \beta_1, \bar{\beta}_2 = a^6 \beta_2, \bar{\beta}_3 = a^8 \beta_3.$$

To satisfy (3.11) we *redefine*

$$(6.18) \quad \pi_i(t) = a \mathcal{P}(t - c_i/2),$$

and set

$$(6.19) \quad G_0(z_1, z_2) = a^4 G(z_1/a, z_2/a),$$

so that

$$(6.20) \quad \overline{G_0(z_1, z_2)} = G_0(\bar{z}_2, \bar{z}_1).$$

In summary we have

PROPOSITION 6.1. *Let  $\Lambda = \mathbf{C}/\Lambda$  have the holomorphic involutions (6.1) intertwined by the anti-holomorphic involution (5.6). Then  $(\Gamma, \rho, \tau_i)$  is realized by the map (6.5), (6.18) onto the quartic curve  $G_0(z_1, z_2) = 0$  given by (6.13), (6.14), (6.19). If the fixed-point set of  $\rho$  is non-empty, then this is the complexification of the real curve  $G_0(z, \bar{z}) = 0$ .*

## 7. A RECTANGULAR LATTICE

We consider the special case of  $\Lambda, \rho, \tau_i$  as given in (5.19), (5.6), (6.1), with

$$(7.1) \quad a = +1, b = 0, \bar{c}_2 = c_1 = c'_1 + ic''_1, c = ic''_1.$$

From (6.16), (6.15) it follows that  $g_2, g_3, \beta$  are real, and  $\beta'$  is purely imaginary. Thus, the coefficients  $\beta_1, \beta_2, \beta_3$  of  $G(z_1, z_2)$  are real. With  $t = t' + it''$ , we have

$$(7.2) \quad FP(\rho) = \{t'' = 0\} \cup \{t'' = \omega''/2\},$$

$$(7.3) \quad \tau_1\{t'' = 0\} = \{t'' = c''_1\}, \tau_1\{t'' = \omega''/2\} = \{t'' = c''_1 + \omega''/2\}.$$

Let us assume that  $0 < c''_1 < \omega''/2$ . Then the torus  $\Lambda$  is divided into four annuli

$$A_1 = \{0 < t'' < c''_1\}, A_2 = \{c''_1 < t'' < \omega''/2\},$$

$$A_3 = \{\omega''/2 < t'' < c''_1 + \omega''/2\}, A_4 = \{c''_1 + \omega''/2 < t'' < \omega''\}.$$

The fixed points of  $\tau_1$  are, by (5.22),

$$(7.4) \quad c_1/2, (c_1 + 1)/2 \in A_1,$$

$$(7.5) \quad (c_1 + i\omega'')/2, (c_1 + 1 + i\omega'')/2 \in A_3.$$