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Artikel: KLYACHKO'S METHODS AND THE SOLUTION OF EQUATIONS OVER TORSION-FREE GROUPS

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Kapitel: 2. The crash theorems

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2. The crash theorems

In this section we prove the crash theorems, which are the main techniques used by Klyachko for his applications.

Some preliminary definitions. Let $p:\mathbf{R}\to S^1$ be the universal covering map of the circle given by $p(t):=e^{2\pi i t}$, $t\in\mathbf{R}$. A function $f:\mathbf{R}\to S^1$ is called *proper, monotone* or *strictly monotone* if its lift $\tilde{f}:\mathbf{R}\to\mathbf{R}$ is proper, monotone or strictly monotone. A monotone function is called *anticlockwise* if its lift is increasing and *clockwise* if its lift is decreasing.

Now let K be a cell complex subdividing the 2-sphere. We shall assume that the 2-sphere has the usual orientation and that each 2-cell is oriented so that its attaching map is anticlockwise.

To help explain the crash theorems we will call each 2-cell a *country*, each 1-cell a *road* and each 0-cell a *junction*. Let $\phi_c: S^1\to\partial c$ denote the attaching map of a country c. A *traffic flow* on K is defined to be a set of proper, monotone, anticlockwise functions $\{f_c:\mathbf{R}\to S^1\}$, one for each country c in K. We will think of $t\in\mathbf{R}$ as a time variable and the point $\kappa_c(t):=\phi_c\circ f_c(t)$ as the position of a car, belonging to c, on the boundary of c at time t. We will say that a car is *on* the road r if it is in the interior of the 1-cell r. The *order* of a junction is the number of ends of roads which are at that junction.

If two or more cars (from neighbouring countries) occupy the same point on a road or the same junction at the same time t, then a *crash* is said to occur at time t.

A *complete crash* occurs if either:

(1) Two cars (from neighbouring countries) occupy the same point on a road at the same time. This is called a *road crash*.

(2) n cars (from all the neighbouring countries) occupy a junction of order n at the same time. Note that it is possible for $n = 1$ so that, paradoxically, a complete crash may involve only one car (crashing into the end of a dead-end road)!

We would like to talk about traffic flows being in "general position". Such a flow would mean that no two cars are at a junction at the same time. There is an obvious notion of a "nearby" flow in which the motion is changed by an amount uniformly less than some positive but small number. However it is important that the result does not increase the number of crashes. The precise statement of the result we need is the following:

LEMMA 2.1. *Suppose for a traffic flow there is an interval of time* $t_0 \leqslant t \leqslant t_1$ *with no complete crashes in some open region* R *of the sphere. Suppose further that cars in* R *are at junctions for just one moment* s, *where* $t_0 < s < t_1$. *Then there is a nearby traffic flow, which is unaltered outside* R *and outside the time interval* $t_0 < t < t_1$, *with no crashes in* R *for* $t_0 \leqslant t \leqslant t_1$ *and such that no two cars in* R *are at junctions at the same time* t *for* $t_0 \leqslant t \leqslant t_1$.

Proof. Suppose there are a number of junctions involved. Then by a small change we can assume that the cars which arrive at the different junctions arrive at different times, without introducing any new crashes, complete or otherwise. So now restrict attention to one junction J in R and consider a small neighbourhood N of J in R. Since by hypothesis there is no complete crash at J for $t_0 \leqslant t \leqslant t_1$ we can assume that the number of cars which meet at J at time s is less than the order of J. A car arriving at a junction turns left. Choose a car κ so that the left turn at J leads to a road whose intersection with N has no car on it. Now hurry κ along so that it arrives ahead of the other cars and completes the turn first. Repeat this process for the remaining cars so that no crashes of any kind occur in N (and note that no new crashes have been introduced outside of N). By adjusting the speeds afterwards we can assume that the flow is unaltered outside of the time interval given. □

THEOREM 2.2. (Basic Crash Theorem.) *Let* K *be a cell decomposition of the 2-sphere with at least one 1-cell. Then any strictly monotone traffic flow on* K *has at least 2 complete crashes at two different places.*

Proof. The hypothesis that K contains at least one 1-cell implies that each 2-cell is attached to one or more 1-cells and that at nearly all times cars will be disjoint and away from junctions. Let t_0 be such a time. Construct an oriented graph Γ, embedded in the 2-sphere, called the *cross-traffic graph,* by the following procedure. (It may be helpful for the reader to consult figure 1 at this point. The cross traffic graph is in heavy print.) For each country c pick some point in the interior as its *capital* C. The capitals will form the vertices of Γ. Suppose that c's car κ_c is on the road forming a common boundary with country c'. Join the capitals C and C' by an edge of Γ oriented from C to C' passing radially outwards in c from C to κ_c and then radially inwards in c' to C'. (It may happen of course that c is c' and so the edge is a loop.)

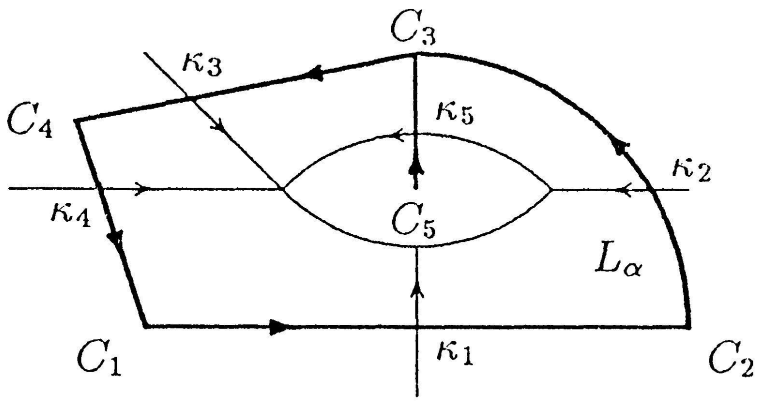

FIGURE 1
The construction of the cross-traffic graph

Notice that Γ is essentially a subgraph of the dual cell subdivision.

Since every capital has an exit path in Γ it follows that Γ will contain coherently oriented simple closed curves. If α is such a simple closed curve let L_α denote the disc with boundary α which is on the left of α when traversed in the direction of orientation. Similarly denote the complementary disc by R_α. Traffic flows *into* L_α as time progresses and *out* of R_α.

Let D be a minimal nested disc amongst the discs L_α and R_α for all loops α of the cross-traffic graph. We shall prove that a complete crash occurs at some time in the interior of D. Since there are at least two such minimal discs (with disjoint interiors) this proves the theorem.

For definiteness assume that $D = L_\alpha$ for some α and watch what happens as time flows forwards. (If $D = R_\alpha$ then we let time flow backwards.) As time proceeds either a road crash occurs or D shrinks upon itself in a continuous fashion until some car inside D comes to a junction at time s say. At this point we have to redefine Γ.

Either there is a complete crash inside D at time s (as required) or by the lemma we can assume that the cars in D arrive at junctions one at a time and we consider the new graph Γ after the first car has passed a junction. There are two possibilities, either the car involved is part of the circuit α or it is not.

If the car is part of α then the corresponding edge breaks the circuit and passes inside D and eventually gives a new circuit defining a new innermost disc inside D. (It can be checked that this is again an L_β for some β.)

If the car is not part of α then either D is still minimal and we proceed or we now have a new minimal disc inside D and we again proceed (in fact the minimality of D implies that the edges of Γ inside D form a forest and it can then be checked that this latter case is impossible, but we shall not need to do this).

Eventually we arrive at a situation where α comprises just one or two edges. In the first case α is a loop around a dead-end junction and a complete crash occurs there and in the second case two cars are approaching each other either on the same road or on two roads with a common junction of order 2 and a complete crash occurs. □

REMARK. In fact it can be seen that there must be infinitely many complete crashes and moreover we can find a subset of these crashes occurring at times t_i, $i \in \mathbf{Z}$ with $t_i < t_{i+1}$ such that, for each i, the crash at time t_i is at a different place to the crash at time t_{i+1}.

TRAFFIC FLOWS WITH STOPS

If we consider traffic flows on a cell decomposition of the 2-sphere which are monotone but not *strictly* monotone, i.e. have *stops*, then it is possible to avoid complete crashes. The following example should make this clear. Consider a neighbourhood of a junction O which we take as the origin and four roads joining O which we take as the coordinate axes. As usual let the increasing direction of the x-coordinate be from west to east and the increasing direction of the y-coordinate be from south to north. Suppose now that there are four cars E, N, S, W approaching O along these roads which in the normal course of events would have a complete crash at O. If stops are allowed then complete crashes can be avoided as follows.

Let E, N and S come to O and crash (incompletely) while W slows down. Now whilst N and S stop at O let E continue south. Now W comes to O and another incomplete crash occurs. The cars can now continue their journey and by adjusting their speeds accordingly can be made to travel as though nothing had happened.

The problem here was that the two stopped cars N and S are next to one another if you ignore E and W.

We shall need to assume that cars which stop at a given vertex do so *each* time they visit that vertex. The following definition for such a traffic flow with stops avoids the problem described above and allows a generalisation of theorem 2.1 to be proved.

DEFINITION. Let v be a junction and let $c_1, \ldots, c_n$ be the countries, listed in anti-clockwise order about v, whose cars $\kappa_1, \ldots, \kappa_n$ actually stop for a positive time at v. Let T_i be the union of the intervals of time that κ_i stops at v. We say that the flow has *separated stops* at v if, for the stopping countries c_i, c_j where $|i - j| = 1 \bmod n$, the unions of intervals T_i and T_j

are disjoint. (Note that under these circumstances, more than one car stops at v and there cannot be a complete crash at v.)

We can now prove a generalisation of the original crash theorem in which cars are allowed to stop.

THEOREM 2.3. (Crash Theorem with Stops.) *Let K be a cell decomposition of the 2-sphere with at least one 1-cell. Then any monotone traffic flow on K with separated stops at each stopping vertex has at least 2 complete crashes at two different places.*

Proof. Let us use the notation developed above. So v is a junction and $c_1, \ldots, c_n$ are the countries, in anticlockwise order about v, whose cars $\kappa_1, \ldots, \kappa_n$ actually stop for a positive time at v. The idea is to change K by blowing up each such junction v to a disc D and defining a strictly monotone traffic flow on a new subdivision K'. This is done as follows. Define the portion of K lying in the interior of D to be a new country. The boundary of D is naturally subdivided into junctions (of order 3) and roads by intersection with the countries adjacent to v. Now collapse to junctions all roads of the boundary of D which are on the boundary of a country whose cars do *not stop* at v (see figure 2).

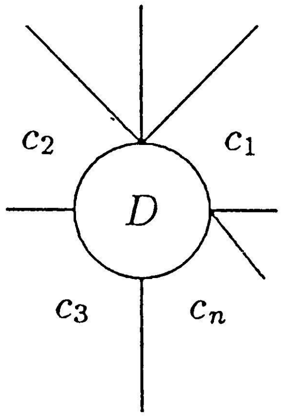

FIGURE 2
The construction of the new cell complex K'

The motion of the original cars which stopped at v can be extended to K' without stops by having them move monotonically along the boundary of D during the time when they originally would have been stopped. The motion of the original cars which do not stop at v is extended to K' in the obvious way.

Now we define the motion of a car κ_D in an anticlockwise manner around the boundary of D. This will be done in such a manner that no complete crashes occur on the boundary of D. We will use the following notation. Let r_i be the road common to the boundary of D and the country with stopping car κ_i. Let the end junctions of r_i be v_i and v_{i+1} in anticlockwise order $i = 1, 2, \dots$. Suppose κ_i is on the road r_i. Then by hypothesis the roads r_{i+1} and r_{i-1} are free of the cars κ_{i+1} and κ_{i-1} respectively (see figure 3).

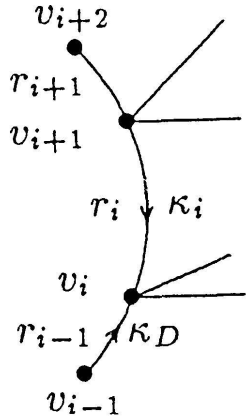

FIGURE 3
The motion of car κ_D

As κ_i traverses r_i from v_{i+1} to v_i let κ_D traverse r_{i-1} from v_{i-1} to v_i. Let the cars meet at v_i at time t. This will not be a complete crash since κ_{i-1} is missing. Again by hypothesis κ_{i+1} will not be at v_{i+2} at time t. Let r be largest such that κ_{i+r} is not at v_{i+r+1} at time t. Then κ_D has enough time to arrive at v_{i+r+1} just as κ_{i+r} does. If there is no such r then let κ_{i+r} be the next car to arrive at D and let κ_D go once round the entire boundary and arrive at v_{i+r+1} just as κ_{i+r} does. Keep repeating this strategem to define the motion of κ_D.

Now we are in a situation corresponding to the first crash theorem and the result is proved. □

3. TWO TRANSVERSALITY LEMMAS

In this section we use transversality (cf. [BRS, F]) to prove the existence of diagrams of van-Kampen type for the two situations that we shall meet in the applications to group theory of the crash theorems (in sections 4, 5 and 6). These lemmas need to be stated very carefully and a failure to do so is one of the major weaknesses in Klyachko's version. The lemmas use the idea of a *corner* of a 2-cell in a cell subdivision K of the 2-sphere. This can be regarded as the (oriented) angle formed by the two adjacent edges meeting at a 0-cell in the boundary of the 2-cell. If all the corners of a 2-cell are labelled by elements of a group, then a word can be read around the 2-cell