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Artikel: KLYACHKO'S METHODS AND THE SOLUTION OF EQUATIONS OVER TORSION-FREE GROUPS

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The t-sequence t^n is interesting because the adjunction problem is already proved (without the torsion-free hypothesis) for words with this t-shape [L]. However the methods discussed here do not extend this result to t-sequences in normal form based on t^n. An example is $t^3t^{-1}t^3t^{-1}$.

Another interesting case is the sequence tt^{-1} which is not amenable. However a simple trick (substitute u^2 for t) makes it suitable. Hence theorem 5.1 implies a solution to the adjunction problem (over torsion-free groups) for words of the form $gtg't^{-1}$. For words of this shape, torsion-free is a necessary condition as the example in the introduction shows!

We do not yet have a simple test for amenability though it is easy from the definition to write down large classes of amenable sequences. However it can be seen that, speaking very roughly, a sequence is amenable unless it is has a uniform slope, like $t^5t^{-3}t^5t^{-3}$ or $t^3t^{-3}t^3t^{-3}$ (slope zero).

6. Further Applications

We give here the other applications from [Kl] of the crash theorems, not covered above.

Theorem 6.1 (Application to free products). *Let A, B be groups and suppose each (cyclic) factor of $u \in A * B - A$ has infinite order. Then the natural homomorphism $A \to \langle A * B \mid [A, u] = 1 \rangle$ is injective.*

Proof. Suppose not. Then the conditions of the first transversality lemma apply and there is a non trivial element $a \in A$ such that $a \in \langle\langle [A, u] \rangle\rangle$. So we have a cell subdivision K of the 2-sphere such that reading round from the base point $*$ for every 2-cell in K spells out the word

$$w(a) = (c_0^{-1} a c_0) c_1 \cdots c_{n-1} (c_n^{-1} a^{-1} c_n) c_{n-1}^{-1} \cdots c_1^{-1}$$

for some $a \in A$, see figure 8. Note that if this 2-cell has the opposite orientation then the word spelt out is $w(a^{-1})$.

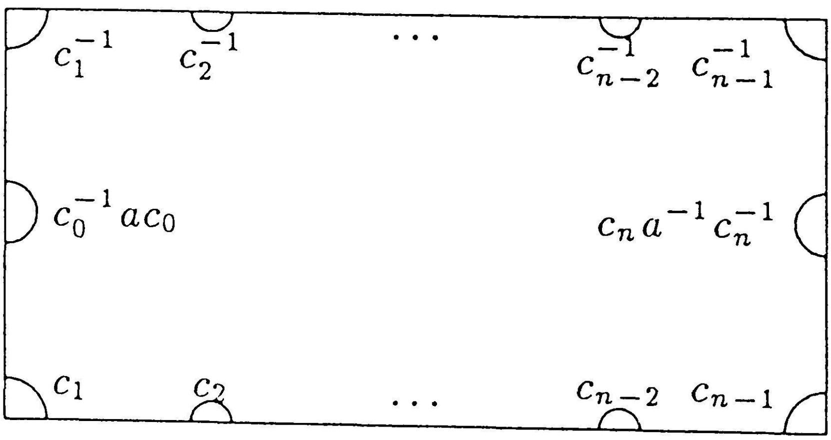

Figure 8
The 2-cell labelled by $w(a)$

Now consider the traffic flow defined as follows. The car associated to a 2-cell starts out from the base point $*$ and proceeds in an anticlockwise manner so that it takes a unit amount of time to reach the next corner. It is clear that any crash must take place at a 0-cell. By the crash theorem there are at least two total crashes so we can assume it takes place at a 0-cell where the angle labels multiply to 1. There are two cases to consider.

If the crash occurs at time 0 mod n then the clockwise labelling around this 0-cell is $c_\alpha^{-1} a_1 c_\alpha, c_\alpha^{-1} a_2 c_\alpha, \ldots, c_\alpha^{-1} a_k c_\alpha$ where $a_1 a_2 \cdots a_k = 1$ and $\alpha = 0$ or $\alpha = n$. A simple calculation shows that the anticlockwise product of the remaining angles of these k 2-cells is 1. So we may simplify the situation by collapsing these k 2-cells to a point.

If the crash occurs at a time $\neq 0$ mod n then the clockwise labelling around this 0-cell is $c_i^k = 1$ for some $0 < i < n$ and some $k > 1$ contradicting the torsion free hypothesis. $\square$

Let H, H' be groups and let $\phi: H \to H'$ be an isomorphism. We shall use the notation h^ϕ to denote the image of $h \in H$ under ϕ. Similarly we shall write $a^b := b^{-1}ab$ for conjugation.

THEOREM 6.2 (Application to HNN extensions). *Let H and H' be two isomorphic subgroups of the group A under the isomorphism $h \to h^\phi$, $h \in H$. Let B be a group and let $w \in A * B - A$ have torsion free factors. Then the natural map*

$$A \to \langle A, B \mid w^{-1}hw = h^\phi, h \in H \rangle$$

is injective.

Proof. Consider the following groups

$$\begin{aligned} A' &= \langle A, t \mid t^{-1}ht = h^\phi, h \in H \rangle, \\ A'' &= \langle A, t, B \mid t^{-1}ht = h^\phi, [a, t^{-1}w] = 1, [t, w] = 1, h \in H, a \in A \rangle \\ &= \langle A', B \mid [a, t^{-1}w] = 1, [t, w] = 1, a \in A \rangle, \\ A''' &= \langle A, B \mid w^{-1}hw = h^\phi, h \in H \rangle. \end{aligned}$$

We can construct the following commuting diagram,

$$\begin{tikzcd} & & A''' \arrow[d, "\gamma"] \\ A \arrow[r, "\alpha"] & A' \arrow[ur, "\delta"] \arrow[r, "\beta"] & A'' \end{tikzcd}$$

where the maps α, β, γ and δ are induced by inclusion. In order for γ to be a well defined homomorphism it is necessary to check that the relation $w^{-1}hw = h^{\phi}, h \in H$ is a consequence of the relations $t^{-1}ht = h^{\phi}, [a, t^{-1}w] = 1, [t, w] = 1, h \in H, a \in A$. But this follows because $w^{-1}hw = w^{-1}tt^{-1}htt^{-1}w = w^{-1}th^{\phi}t^{-1}w = h^{\phi}$. Now α is injective because A' is an HNN extension of A (see [DD, p. 33] or [Se, p. 9]) and β is injective because of theorem 6.1. So δ is injective and this proves the theorem. □

THEOREM 6.3. *Let*

$$(*) \qquad u_i(t) = 1, \; i \in I$$

be a set of equations over the group A *where the exponent sum of* t *in each* $u_i(t)$ *is zero. Suppose* $w = w(t) \in A * \langle t \rangle - A$ *and the factors of* w *are all torsion free. Then the set of equations*

$$(**) \qquad u_i(w(t)) = 1, \; i \in I$$

has a solution over A *if and only if the set* $(*)$ *has a solution over* A.

Proof. Let $w(t) = at$ where $a \in A$ has infinite order. Then a solution x for $u_i(w(t)) = 1$ defines a solution at for $(*)$.

Conversely suppose $x \in A'$ is a solution of the set of equations $\{u_i(t) = 1 \mid i \in I\}$. Let G be the subgroup of A' generated by

$$\{x^{-n}ax^n \mid a \in A, n \in \mathbf{Z}\}.$$

Then A is a subgroup of G and G is a subgroup of

$$H = \langle G, t \mid w^{-1}gw = g^{\phi}, g \in G \rangle$$

where $g^{\phi} = x^{-1}gx$ by theorem 6.2. Because of the exponent sum condition $u_i(w) = 1, i \in I$. □

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