

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID
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Kapitel: 7. Centralisers in SB_n
DOI: <https://doi.org/10.5169/seals-87872>

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7. CENTRALISERS IN SB_n

7.1 THEOREM. *For a singular braid $x \in SB_n$ the following are equivalent:*

- (a) $\sigma_j x = x \sigma_k$;
- (b) $\sigma_j^r x = x \sigma_k^r$, for some nonzero integer r ;
- (c) $\sigma_j^r x = x \sigma_k^r$, for every integer r ;
- (d) $\tau_j x = x \tau_k$;
- (e) $\tau_j^r x = x \tau_k^r$, for some nonzero integer r ;
- (f) x has a (possibly singular) (j, k) -band.

Proof. We only show (b) implies (f). The other implications are quite clear. The term “band” will include the possibility of a singular band. Suppose $\sigma_j^r x = x \sigma_k^r$. Then we have that $\sigma_j^{2r} x = x \sigma_k^{2r}$. Assume $x = \beta \tau_i y$, where β is a braid; in other words τ_i is the first singular generator appearing in x . Then we have $\beta^{-1} \sigma_j^{2r} \beta \tau_i y = \tau_i y \sigma_k^{2r}$. Recall that isotopy, or the extended Reidemeister moves for singular braids, do not change the order of singular generators on the same strings. Since $\beta^{-1} \sigma_j^{2r} \beta$ is a pure braid, the τ_i in $\tau_i y \sigma_k^{2r}$ corresponds under some homeomorphism, to the τ_i in $\beta^{-1} \sigma_j^{2r} \beta \tau_i y$. Hence the image, under that homeomorphism, of the trivial singular band near the first τ_i provides a band for $\beta^{-1} \sigma_j^{2r} \beta$. Therefore, τ_i commutes with $\beta^{-1} \sigma_j^{2r} \beta$. It follows that $\tau_i \beta^{-1} \sigma_j^{2r} \beta y = \tau_i y \sigma_k^{2r}$. By Proposition 5.1, we have $\beta^{-1} \sigma_j^{2r} \beta y = y \sigma_k^{2r}$, i.e. $\sigma_j^{2r} \beta y = \beta y \sigma_k^{2r}$. By induction, βy has a (j, k) -band. Since τ_i commutes with $\beta^{-1} \sigma_j^{2r} \beta$, so does σ_i , thus we have $\beta^{-1} \sigma_j^{2r} \beta \sigma_i y = \sigma_i \beta^{-1} \sigma_j^{2r} \beta y = \sigma_i y \sigma_k^{2r}$, i.e. $\sigma_j^{2r} \beta \sigma_i y = \beta \sigma_i y \sigma_k^{2r}$. It follows from induction assumption that $\beta \sigma_i y$ has a (j, k) -band. Since both βy and $\beta \sigma_i y$ have a (j, k) -band, we can use the argument of Lemma 6.4 to conclude that $x = \beta \tau_i y$ has a (j, k) -band. \square

The above theorem allows us to identify monoid centralisers in SB_n . Notice that SB_2 is abelian. On the other hand, for $n \geq 3$, any singular braid with a singularity involving strings labelled, say, j and k , $j < k$, could not possibly commute with τ_k , as any (singular) band from $[k, k+1] \times 0$ to $[k, k+1] \times 1$ would have a forbidden intersection with the j string, see Figure 7. Therefore for $n \geq 3$, only *nonsingular* braids are central. We will conclude with two applications whose proofs, at this point, can safely be left to the reader.

7.2 THEOREM. *The centre of SB_n is all of SB_n for $n = 2$. But in case $n \geq 3$ it is the same as the (infinite cyclic) centre of $B_n \subset SB_n$, generated by Δ^2 . \square*

7.3 THEOREM. *Under the natural inclusion, the centraliser of SB_r in SB_n , $r \leq n$, is generated as a monoid by the generators (see Theorem 4.4) of $C(r, n)$:*

$$\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_{n-1}, A_{r+1}, \dots, A_n, C,$$

together with their inverses and the singular generators:

$$\begin{aligned} & \tau_{r+1}, \dots, \tau_{n-1} && \text{if } r \geq 3, \text{ or} \\ & \tau_1, \tau_3, \tau_4, \dots, \tau_{n-1} && \text{if } r = 2. \end{aligned}$$

\square

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