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Artikel: CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID

Autor: Fenn, Roger / Zhu, Jun

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7. Centralisers in SB_n

7.1 Theorem. *For a singular braid* $x \in SB_n$ *the following are equivalent:*

(a) $\sigma_j x = x\sigma_k$;

(b) $\sigma_j^r x = x\sigma_k^r$, *for some nonzero integer* r;

(c) $\sigma_j^r x = x\sigma_k^r$, *for every integer* r;

(d) $\tau_j x = x\tau_k$;

(e) $\tau_j^r x = x\tau_k^r$, *for some nonzero integer* r;

(f) x *has a (possibly singular)* (j, k)*-band.*

Proof. We only show (b) implies (f). The other implications are quite clear. The term "band" will include the possibility of a singular band. Suppose $\sigma_j^r x = x\sigma_k^r$. Then we have that $\sigma_j^{2r} x = x\sigma_k^{2r}$. Assume $x = \beta\tau_i y$, where β is a braid; in other words τ_i is the first singular generator appearing in x. Then we have $\beta^{-1}\sigma_j^{2r}\beta\tau_i y = \tau_i y\sigma_k^{2r}$. Recall that isotopy, or the extended Reidemeister moves for singular braids, do not change the order of singular generators on the same strings. Since $\beta^{-1}\sigma_j^{2r}\beta$ is a pure braid, the τ_i in $\tau_i y\sigma_k^{2r}$ corresponds under some homeomorphism, to the τ_i in $\beta^{-1}\sigma_j^{2r}\beta\tau_i y$. Hence the image, under that homeomorphism, of the trivial singular band near the first τ_i provides a band for $\beta^{-1}\sigma_j^{2r}\beta$. Therefore, τ_i commutes with $\beta^{-1}\sigma_j^{2r}\beta$. It follows that $\tau_i\beta^{-1}\sigma_j^{2r}\beta y = \tau_i y\sigma_k^{2r}$. By Proposition 5.1, we have $\beta^{-1}\sigma_j^{2r}\beta y = y\sigma_k^{2r}$, i.e. $\sigma_j^{2r}\beta y = \beta y\sigma_k^{2r}$. By induction, βy has a (j, k)-band. Since τ_i commutes with $\beta^{-1}\sigma_j^{2r}\beta$, so does σ_i, thus we have $\beta^{-1}\sigma_j^{2r}\beta\sigma_i y = \sigma_i\beta^{-1}\sigma_j^{2r}\beta y = \sigma_i y\sigma_k^{2r}$, i.e. $\sigma_j^{2r}\beta\sigma_i y = \beta\sigma_i y\sigma_k^{2r}$. It follows from induction assumption that $\beta\sigma_i y$ has a (j, k)-band. Since both βy and $\beta\sigma_i y$ have a (j, k)-band, we can use the argument of Lemma 6.4 to conclude that $x = \beta\tau_i y$ has a (j, k)-band. □

The above theorem allows us to identify monoid centralisers in SB_n. Notice that SB_2 is abelian. On the other hand, for $n \geqslant 3$, any singular braid with a singularity involving strings labelled, say, j and k, $j < k$, could not possibly commute with τ_k, as any (singular) band from $[k, k+1] \times 0$ to $[k, k+1] \times 1$ would have a forbidden intersection with the j string, see Figure 7. Therefore for $n \geqslant 3$, only *nonsingular* braids are central. We will conclude with two applications whose proofs, at this point, can safely be left to the reader.

7.2 THEOREM. *The centre of* SB_n *is all of* SB_n *for* $n = 2$. *But in case* $n \geqslant 3$ *it is the same as the (infinite cyclic) centre of* $B_n \subset SB_n$, *generated by* Δ^2. □

7.3 THEOREM. *Under the natural inclusion, the centraliser of* SB_r *in* SB_n, $r \leqslant n$, *is generated as a monoid by the generators (see Theorem 4.4) of* $C(r, n)$*:*

$$\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_{n-1}, A_{r+1}, \ldots, A_n, C,$$

together with their inverses and the singular generators:

$$\tau_{r+1}, \ldots, \tau_{n-1} \quad \textit{if } r \geqslant 3, \textit{ or}$$
$$\tau_1, \tau_3, \tau_4, \ldots, \tau_{n-1} \quad \textit{if } r = 2\,.$$
□

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